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ON THE AVERAGE OF THE GENERALIZED TOTIENT FUNCTION OVER POLYNOMIAL SEQUENCES

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Let $\phi_F^{k,\eta}(n) = \phi^{k,\eta}_{f_1,u_1;\, f_2,\, u_2;\, \cdots;\, f_s,\, u_s}(n)$ be the generalized totient function; here $f_1 = f_1(x), \cdots, f_s = f_s(x)$ are integer coefficient polynomials of positive degrees, $\eta = \eta(n)$ an arithmetical function, and $u_1, u_2, \cdots u_s$ and k are positive integers and $t = u_1 + u_2 + \cdots + u_s$. In this paper, asymptotic formulas for

$$\sum_{\substack{n \leqslant x \\ f(n) \neq 0}} \phi_F^{k,\eta}(|f(n)|) \text{ and } \sum_{\substack{n \leqslant x \\ f(n) \neq 0}} \frac{\phi_F^{k,\eta}(|f(n)|)}{f(n)^{kt}}$$

$f = f(x)$ being an integer coefficient polynomial of positive degree h and with positive leading coefficient a_h, are obtained under the assumption $\eta(n) = O(n^{\epsilon})$, $0 \leqslant \epsilon < 1/h$.

1. INTRODUCTION

Let $\phi(n)$ be the Euler totient function and $f(x)$ an arbitrary integer coefficient polynomial of positive degree h and positive leading coefficient a_h. Shapiro[7] proved in his book (p. 175-80) that if $f(x)$ has no multiple roots and $f(n) > 0$ for $n \geqslant 1$, then

$$\sum_{n \leqslant x} \frac{\phi(f(n))}{f(n)} = ax + O(\log^h x) \qquad \text{...(1.1)}$$

and

$$\sum_{n \leqslant x} \phi(f(n)) = \frac{a\, a_h}{h+1} x^{h+1} + O(x^h \log^h x) \qquad \text{...(1.2)}$$

where a is given by

$$a = \sum_{n=1}^{\infty} \frac{\mu(n)\, \rho_f(n)}{n^2}. \qquad \text{...(1.3)}$$

In (1.3) $\mu(n)$ is the Mobius function and $\rho_f(n)$ is the number of incongruent solutions mod n of

$$f(x) \equiv O \pmod{n}. \qquad \ldots(1.4)$$

The purpose of this paper is to extend the above results to the generalized totient function $\phi_F^{k,\eta}(n)$ introduced in Chidambraswamy[2].

We recall that, given a integer coefficient polynomials $f_i = f_i(x)$ of positive degrees for $1 \leqslant i \leqslant s$, the arithmetical function $\eta = \eta(n)$, the positive integers $u_1, u_2, \ldots u_s$ with $t = u_1 + u_2 + \ldots + u_s$, and the positive integer k, the arithmetical function $\phi^{k,\eta}_{f_1,u_1; f_2,u_2; \ldots; f_s,u_s}(n) = \phi_F^{k,\eta}(n)$ is defined by

$$\phi_F^{k,\eta}(n) = n^{kt} \sum_{d \mid n} \frac{\mu_F^{k,\eta}(d)}{d^{kt}} \qquad \ldots(1.5)$$

where

$$\mu_F^{k,\eta}(n) = \mu(n)\,\eta(n)\,Q_F(n^k) \qquad \ldots(1.6)$$

$$Q_F(n) = \rho_{f_1}^{u_1}(n)\,\rho_{f_2}^{u_2}(n) \ldots \rho_{f_s}^{u_s}(n) \qquad \ldots(1.7)$$

it being understood that for any arithmetical function $\lambda(n)$, $\lambda^m(n = (\lambda(n))^m$.

The function $\phi^{k,I}_{f_1 u_1}(n)$ $(s = 1)$, $I = I(n)$ being the function defined as $I(n) = 1$ for all n, has been studied in Chidambaraswamy[3] and this function includes, as special cases, $\phi(n)$ and its various generalizations. In fact,

$$\phi^{1,I}_{x,1}(n) = \phi(n), \quad \phi^{1,I}_{x,u_1}(n) = J_{u_1}(n),$$

$$\phi^{k,I}_{x,1}(n) = \phi_k(n), \quad \phi^{1,I}_{f_1,1}(n) = \phi_{f_1}(n),$$

where $J_{u_1}(n)$ is the Jordan totient function of order u_1 Dickson[4] and $\phi_k(n)$ and $\phi_{f_1}(n)$ are respectively the generalizations of $\phi(n)$ introduced by Cohen[5] and Menon[6]. The function $\phi^{1,\mu_u}_{x,1}(n) = \phi_{\mu_u}(n)$ is introduced by Venkataraman and Sivaramakrishnan[8]; here the function $\mu_u = \mu_u(n)$ is definedae $\mu_u(n) = \exp(i\pi w(n)/_u)$ or zero according as n is or is not squarefree, $w(n)$ being the number of distiact prime factors of n.

Let $B = DD_1 D_2 \ldots D_s$ where D is the g.c d. of the coefficients of f and D_i for $1 \leqslant i \leqslant s$ is the g.c.d. of the coefficient of f_i and let P be the largest prime factor of B in case $B > \eta$. Let H be the maximum of the degrees of $f(x), f_1(x), \ldots, f_s(x)$ and let C be taken as$H$ or max $\{H, p\}$ according as $b = 1$,or $B < 1$. Then it is not hard to see that

$$\rho_{f_i}(p) \leqslant C, \; 1 \leqslant i \leqslant s; \; \rho_f(p) \leqslant C \qquad \ldots(1.8)$$

for all primes p. We shall write L for C^{t+1}. We shall prove the following theorems.

Theorem 1—Let $f(x)$ be an arbitrary integer coefficient polynomial of positive degree h and leading coefficient $a_h > 0$. Then, if $\eta(n) = O(n)$, $0 \leqslant \epsilon < 1/h$,

$$\sum_{\substack{n \leqslant x \\ f(n) \neq 0}} \frac{\phi_F^{k,\eta}(|f(n)|)}{|f(n)|^{kt}} = Ax + E(x) \qquad \ldots(1.9)$$

where

$$A = \sum_{n=1}^{\infty} \frac{\mu_F^{k,\eta}(n)\, \rho_f(n)}{n^{kt+1}} \qquad \ldots(1.0)$$

and

$$E(x) = \begin{cases} O(1), & t > 1 \\ x^{h\epsilon} (\log x)^1 & t = 1. \end{cases} \qquad \ldots(1.11)$$

Theorem 2— Under the hypothesis of Theorem 1,

$$\sum_{\substack{n \leqslant x \\ f(n) \neq 0}} \phi_F^{k,\eta}(|f(n)|) = A\, a_h^{kt} \frac{x^{hkt+1}}{hkt+1} + E_1(x) \qquad \ldots(1.12)$$

where

$$E_1(x) = \begin{cases} O(x^{hkt}), & t > 1 \\ O(x^{(kt+\epsilon)h} (\log x)^L), & t = 1. \end{cases} \qquad \ldots(1.13)$$

2. Proofs of the Theorems

Lemma 1—If $\eta(n) = O(n^\epsilon)$, $0 \leqslant \epsilon < 1$,

the series $A = \displaystyle\sum_{n=1}^{\infty} \frac{\mu_F^{k,\eta}(n)\, \rho_f(n)}{n^{kt+1}}$ converges absolutely.

Proof : Since for arbitrary integer coefficient polynomial $g = g(x)$, $\rho_g(n)$ is a multiplicative function of n, i. e. $\rho_g(mn) = \rho_g(m) = \rho_g(n)$ whenever $(m, n) = 1$, and since, for prime powers p^α, $\rho_g(p^\alpha) \leqslant p^{\alpha-1} \rho_g(p)$

we have, by (1.8), for square free integers n

$$\rho_f(n^k) \leqslant n^{k-1} C^{w(n)}, \ 1 \leqslant i \leqslant s; \ \rho_f(n) \leqslant C^{v(n)} \qquad \ldots(2.1)$$

Thus for such integers n by (1.7) and (2.1)

$$Q_F(n^k)\,\rho_f(n) \leqslant n^{(k-1)t}\,(C^{t+1})^{w(n)} = n^{(k-1)t}\,L^{w(n)}. \qquad \ldots(2.2)$$

Now, since $L^{w(n)} = 2^{w(n)\frac{\log L}{\log 2}} \leqslant \tau(n)^{\frac{\log L}{\log 2}}$, $\tau(n)$ being the number of positive divisors of n and since $\tau(n) = 0\,(n^{\theta})$ for every $\theta > 0$ we have for every $\theta > 0$ by (2.2) and (1.6)

$$\frac{|\,\mu_F^{k,\eta}(n)\,|\,\rho_f(n)}{n^{kt+1}} = O\left(\frac{1}{n^{t+1}-(\epsilon+\theta)}\right)$$

and the lemma follows since $t \geqslant 1$.

Lemma 2—For each positive integer m,

$$\sum_{n\leqslant x} m^{w(n)} = O\,(x\,(\log x)^{m-1}).$$

PROOF : We use induction on m. The result is obvious for $m = 1$. Assuming the result for m and observing that

$$(m+1)^{w(n)} = \sum_{d|n} \mu^2(d) m^{w(d)} \leqslant \sum_{d|n} m^{w(d)},$$

we have

$$\sum_{n\leqslant x} (m+1)^{w(n)} \leqslant \sum_{d\delta<x} m^{w(d)} \leqslant x \sum_{d\leqslant x} \frac{m^{w(d)}}{d}$$

$= O\,(x\,(\log x)^m$, where in the last step we used inductive hypothesis and partial summation (Theorem 4.2 of Apostol[1]).

Lemma 3—If $\bar{\rho}_f(n) = \max\{1, \rho_f(n)\}$ and $\eta(n) = O\,(n^{\epsilon})$, $0 \leqslant \epsilon < 1$,

$$\sum_{n>x} \frac{|\,\mu_F^{k,\eta}(n)\,|\,\bar{\rho}_f(n)}{n^{kt+1}} = O\,(x^{-t+\epsilon}\,(\log x)^{L-1})) \qquad \ldots(2.3)$$

$$\sum_{n\leqslant x} \frac{|\,\mu_F^{k,\eta}(n)\,|\,\bar{\rho}_f(n)}{n^{kt}} = \begin{cases} O\,(x^{\epsilon}\,(\log x)^{L}), & t = 1, \\ O\,((1), & t > 1. \end{cases} \qquad \ldots(2.4)$$

PROOF : Since for square free integers n $\rho_f(n) \leqslant C^{w(n)}$, we have for such integers n $\bar{\rho}_f(n) \leqslant C^{w(n)}$. Hence, by (2.2) (with $\bar{\rho}_f(n)$ in place of $\rho_f(n)$) and (1.6) we get

$$|\,\mu_F^{k,\eta}(n)\,|\;\bar{\rho}_f(n) = O\,(n^{(k-1)t+\epsilon}\,L^{w(n)}).$$

Now an application of Lemma 2 and partial summation give Lemma 3.

Proof of Theorem 1—Let $X > 0$ be so chosen that (1) $f(x) > 0$ and increasing and (2)

$$\frac{2\,a_h}{3}\,x^h < f(x) < \frac{4\,a_h}{3}\,x^h, \text{ for } x > X. \qquad \ldots(2.5)$$

We have,

$$\sum_{\substack{n\leqslant x\\ f(n)\neq 0}} \frac{\phi_F^{k,\eta}(\,|\,f(n)\,|\,)}{|\,f(n)\,|^{kt}} = \sum_{X<n\leqslant x} \frac{\phi_F^{k,\eta}(f(\eta))}{f(n)^{kt}} + O(1)$$

and this by (1.5) is

$$= \sum_{X<n\leqslant x}\;\sum_{d|f(n)} \frac{\mu_F^{k,\eta}(d)}{d^{kt}} + O(1)$$

$$= \sum_{d\leqslant f(x)} \frac{\mu_F^{k,\eta}(d)}{d^{kt}} \Bigg(\sum_{\substack{X<n\leqslant x\\ f(n)\equiv O \,(\mathrm{mod}\, d)}} 1 \Bigg) + O(1)$$

$$= \sum_{d\leqslant f(X)} \frac{\mu_F^{k,\eta}(d)}{d^{kt}} \{(\frac{x}{d})\,\rho_f(d) + O(\bar{\rho}_f\;(d)) + O(1)\} + O(1)$$

and this by Lemma 1 is

$$= Ax + O\Bigg(\sum_{d>f(x)} \frac{|\,\mu_F^{k,\eta}(d)\,|\,\bar{\rho}_f\,(d)}{d^{kt+1}} \Bigg)$$

$$+ O\Bigg(\sum_{d\leqslant f(x)} \frac{|\,\mu_F^{k,\eta}(d)\,|\,\bar{\rho}_f\,(d)\,|}{d^{kt}} \Bigg) + O\,)1)$$

$$= Ax + O_1 + O_2 + O(1), \text{ say.}$$

Now, by (2.5) and Lemma 3, we get

$$O_1 = O\Bigg(\sum_{n>\frac{2a_h}{3}x^h} \frac{|\,\mu_F^{k,\eta}(n)\,|\,\bar{\rho}_f\,(n)}{n^{kt+1}} \Bigg)$$

$$= O\;(x^{-h(t-\epsilon)}\,(\log x)^{L-1})$$

and similarly,

$$O_2 = O\,(x^{ht}\,(\log x)^L) \text{ or } O\,(1)$$

according as $t = 1$ or $t > 1$ and the proof of the Theorem 1 is complete.

Proof of Theorem 2—Let the positive integer X be chosen that $f(x) > 0$ and increasing on $[X, \infty)$. Writing $S(x)$ for the L.H.S of (1.9), we have

$$\sum_{\substack{n \leqslant x \\ f(n) \neq 0}} \phi_F^{k,n}(\,|\,f(n)\,|\,) = \sum_{X < n \leqslant x} \phi_F^{k,n}\,(f(n)) + O\,(1)$$

$$= \sum_{m=X+1}^{[x]} f(m)^{kt}\,\{S\,(m) - S\,(m-1)\} + O\,(1)$$

$$= -\sum_{m=X-1}^{[X]-1} S\,(m)\,\{\,f\,(m+1)^{kt} - f\,(m)^{kt}\}$$

$$+\; S\,([x])\,f\,([x])^{kt} + O\,(1). \qquad \ldots(2.6)$$

Now, since

$$S\,([x] = A\,[x] + E\,([x])$$

$$= Ax + O\,(\,|\,E([x]\,|\,),$$

and

$$f\,([x])^{kt} = a_h^{kt}\; x^{hkt} + O\,(x^{hkt-1})$$

we have

$$S\,([x])\,f\,([x])^{kt} = A\,a_h^{kt}\;\; x^{hkt+1} + O\,(\,|\,E\,([x])\,|\,x^{hkt}). \qquad \ldots(2.7)$$

Further, we have

$$\sum_{m=X+1}^{[X]-1} S\,(m)\,\{\,f\,(m+1)^{kt} - f\,(m)^{kt}\}$$

$$= \sum_{m=X+1}^{[X]-1} \{Am + E(m)\}\,\{\,f\,(m+1)^{kt} - f\,(m)^{k'}\}$$

$$= A \sum_{m=X+1}^{[X]-1} m\,\{hkt\,a_h^{kt}\;\; m^{hkt-1} + O\,(m^{hkt-2})\}$$

$$+\; O\,(\sum_{m=X+1}^{[X]-1} |\,E\,(m)\,|\,\{\,f\,(m+1)^{kt} - f\,(m)^{kt}\}). \qquad \ldots(2.8)$$

Now, since

$$\sum_{m=X+1}^{[X]-1} | E(m) | \{ f(m+1)^{kt} - f(m)^{kt} \} = \begin{cases} O(x^{h(kt+\epsilon)} (\log x)^L), \; t = 1 \\ O(x^{hkt}), \; t > 1 \end{cases} \quad \ldots(2.9)$$

and since for $u \geqslant 1$

$$\sum_{n \leqslant x} n^u = \frac{x^{u+1}}{u+1} + O(x^u).$$

Theorem 2 follows from (2.6), (2.7), (2.8) and (2.9).

Remark : In the special case of the Euler totient function $\phi(n)$ (in fact, in the case of $\phi_k(n)$, $J_k(n)$ and $\phi_{p_u}(n)$) we have $s = 1, f_1(x) = x, \epsilon = 0, Q_F(n) = 1$. If $f(x)$ is primitive, we can take $C = h$ and the error terms in Theorems 1 and 2 will be the same as those of (1.1) and (1.2).

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A NOTE ON JORDAN'S TOTIENT FUNCTION

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Jordan's totient function J_k is a genaralization of the Euler's totient function ϕ. In this paper, the norm of this Jordan function and that of its conjugate have been obtained. Some interesting congruence properties of J_k have also been obtained.

A generalization of the famous Euler's totient function is the Jordan's totient function defined by

$$J_k(n) = n^k \prod_{p \mid n} (1 - p^{-k}).$$

We define the conjugate of this function as $\bar{J_k}(n) = n^k \prod_{p \mid n} (1 + P^{-k})$ which is introduced as generalization of Dedekind ψ— function by Suryanarayana[3]. Following the techniques employed by Menon[1] and Sivaramakrishnan[2], we have obtained in this paper the norms of the functions $J_k(n)$, $\bar{J_k}(n)$, $\phi(n)$, $\bar{\phi}(n)$, $J_k^{-1}(n)$, $\phi^{-1}(n)$ and established some interesting relationship among them.

We have also obtained some interesting congruence properties of $J_k(n)$.

Definition 1 — The norm of a multiplicative function f is the arithmetic function $M(f)$ defined by $M(f)(n) = \sum_{d \mid n^2} f(n^2/d)\, \lambda(d) f(d)$ for all n. The norm defined above has been proved to be a multiplicative function and further, if f, g are multiplicative then $M(f * g) = M(f) * M(g)$.

Theorem 1— $M(J_k) = M(\bar{J_k})$.

PROOF : We know that $J_k = \mu * N_k$, where μ is Mobius function and N_k is an arithmetic function defined by $N_k(n) = n^k$.

$M(J_k) = M(\mu) * M(N_k)$, since μ and N_k are multiplicative. If p is any prime and $\alpha > 0$,

$$M(J_k)(P^\alpha) = \sum_{d \mid P^\alpha} M(\mu)(d)\, M(N_k)(p^\alpha/d)$$

(*equation continued on p.* 1162)

$$= M(\mu)(1)\, M(N_k)(p^\alpha) + M(\mu)(p)\, M(N_k)(P^{\alpha-1})$$

$$+ \ldots + M(\mu)(P^\alpha)\, M(N_k)(1) \qquad \ldots(1)$$

$$M(\mu)(1) = 1$$

$$M(N_k)(P^\alpha) = P^{2\alpha k}$$

$$M(\mu)(P) = (-1)$$

and

$$M(N_k)(P^{\alpha-1}) = P^{2(\alpha-1)k}$$

$$M(\mu)(p^\alpha) = 0 \text{ for } \alpha > 1.$$

Hence (1) reduces to

$$M(J_k)(P^\alpha) = P^{2\alpha k} - p^{2(\alpha-1)k}$$

$$= J_{2k}(P^\alpha).$$

Thus if $\quad n = \prod_{i=1}^{\gamma} P_i^{\alpha_i}$

then

$$M(J_k)\left(\prod_{i=1}^{\gamma} P_i^{\alpha_i}\right) = \prod_{i=1}^{\gamma} M(J_k)\left(P_i^{\alpha_i}\right)$$

$$= \prod_{i=1}^{\gamma} J_{2k}\left(P_i^{\alpha_i}\right) = J_{2k}\left(\prod_{i=1}^{\gamma} P_i^{\alpha_i}\right)$$

$$= J_{2k}(n).$$

$$\bar{J_k} = \bar{\lambda}^1 * N_k.$$

For any prime p, and $\alpha > 0$,

$$M(\bar{J_k})(P^\alpha) = \sum_{d \mid p^\alpha} M(\lambda^{-1})(d)\, M(N_k)(P^\alpha/d)$$

$$= M(\lambda^{-1})(1)\, M(N_k)(P^\alpha) + M(\lambda^{-1})(P)\, M(N_k)(P^{\alpha-1})$$

$$+ \ldots + M(\lambda^{-1})(P^\alpha)\, M(N_k)(1).$$

$$M(\lambda^{-1})(1) = 1.$$

$$M(N_k)(P^\alpha) = P^{2\alpha k}.$$

$$M(\lambda^{-1})(P) = (-1).$$

$$M(N_k)(P^{\alpha-1}) = P^{2(\alpha-1)k}.$$

It can be easily seen that $M(\lambda^{-1})(P^\alpha) = 0$ for $\alpha > 1$.

Thus

$$M(\bar{J_k})(P^{\alpha}) = P^{2\alpha k} - P^{2(\alpha-1)k}$$
$$= J_{2k}(P^{\alpha}).$$

Consequently, $M(J_k)(n) = \bar{J}_{2k}(n)$, thus we see that $M(\bar{J}_k) = M(J_k) = J_{2k}$.

The following theorem is an immediate consequence.

Theorem 2— $M(\phi) = M(\bar{\phi}) = J_2$, | where $\bar{\phi} = \bar{J_1}$ | .

Theorem 3— (1) $M(J_k^{-1})(n) = J_{2k}^{-1}(n)$, (2) $M(\phi^{-1})(n) = J_2^{-1}(n)$.

PROOF : (1) we have $J_k^{-1} = \mu N_k * u$, so that

$$M\left(J_k^{-1}\right) = M(\mu N_k) * M(u).$$

If P is any prime, and $\alpha > 0$,

then

$$M\left(J_k^{-1}\right)(p^{\alpha}) = \sum_{d \mid p^{\alpha}} M(\mu N_k)(d)\, M(u)(P^{\alpha}/d).$$

Since

$$M(\mu N_k)(1) = 1$$

$$M(u)(p^{\alpha}) = 1$$

$$M(\mu N_k)(p) = (-p^{2k}),$$

$$M(u)(p^{\alpha-1}) = 1$$

and

$M(\mu N_k)(p^{\alpha}) = 0$, for $\alpha > 1$, it follows that

$$M\left(J_k^{-1}\right)(p^{\alpha}) = 1 - p^{2k} = J_{2k}^{-1}(p^{\alpha}).$$

Also J_k being a multiplicative function implies that J_k^{-1} is also a multiplicative function.

Hence we obtain $M\left(J_k^{-1}\right)(n) = J_{2k}^{-1}(n)$.

Simillarly, it can be easily deduced that $M(\phi^{-1})(n) = J_2^{-1}(n)$.

Theorem 4— $J_k(n)$ is even if and only if $n \geqslant 3$.

PROOF : $J_k(1) = 1$

$$J_k(2) = 2^k - 1 \equiv 1 \pmod 2.$$

Hence $J_k(n)$ is odd if $n \leqslant 2$.

Let $n > 2$ and let

$$n = \prod_{i=1}^{\gamma} p_i^{\alpha_i}$$

For any odd prime p, we have,

$$J_k(p^{\alpha}) = p^{(\alpha-1)k}(p^k - 1).$$

Now since $p \equiv 1 \pmod 2$, $p^k \equiv 1 \pmod 2$.

Hence $J_k(n) \equiv O \pmod 2$ if n has some odd prime factor. Also, if n has no odd prime factor i.e., if $n = 2^{\alpha}$,

$$J_k(2^{\alpha}) = 2^{(\alpha-1)k}(2^k - 1) \equiv O \pmod 2.$$

It follows therefore that for $n > 2$, $J_k(n)$ is even.

Theorem 5— $J_k(n) \equiv O \pmod 3$ if and only if atleast one of the following three conditions is true :

(1) $3^3/n$

(2) $P_i \equiv 1 \pmod 3$ for some i

(3) $P_i \equiv 2 \pmod 3$, for some i, with α_i even

where P_i is some prime factor of n.

PROOF : Let $n = \prod_{i=1}^{\gamma} p_i^{\alpha_i}$.

If $3^2 \mid n$, then $J_k(n) \equiv O \pmod 3$.

If $P_i \equiv 1 \pmod 3$ for some $P_i \mid n$, then $P_i^k \equiv 1 \pmod 3$.

Hence

$$J_k\left(P_i^{\alpha_i}\right) \equiv P_i^{(\alpha_i-1)k}\left(P_i^k - 1\right) \equiv O \pmod 3$$

If $P_i \equiv 2 \pmod 3$ then $P_i^k \equiv 1 \pmod 3$, if k is even, so that $J_k\left(P_i^{\alpha_i}\right) \equiv O \pmod 3$ in this case also.

This proves the if part.

Conversely, $J_k(n) \equiv O \pmod 3$ implies

$$\prod_{i=1}^{\gamma} J_k\left(P_i^{\alpha_i}\right) \equiv O \pmod 3 \text{ (since } J_k(n) \text{ is multiplicative).}$$

Hence

$$J_k\left(P_i^{\alpha_i}\right) \equiv O \pmod 3 \text{ for some } P_i \mid n.$$

This means $3 \mid P_i$ or $3 \mid (P_i - 1)$ or $3 \mid (1 + P + \ldots + P^{k-1})$, thus $P_i \equiv O \pmod 3 \Rightarrow 3^2 \mid n$ as $3 \nmid J_k(3)$ or $P_i \equiv 1 \pmod 3$ or $3 \mid (1 + P + \ldots + P^{k-1})$. If $3 \nmid P_i$ and $3 \nmid (P_i - 1)$ then $3 \mid (1 + P_i)$ and $3 \mid (1 + P + \ldots + P^{k-1})$ and this is possible only when K is even. Hence if $3 \nmid P_i$, $3 \nmid (P_i - 1)$ then $3 \mid (P_i + 1)$ and K is even.

Hence the Theorem

We ha ve the following corollary at once from the above two theorems.

Corollary 6— If $K > 1$, $J_k(n) \equiv O \pmod 6$ if atleast one of the following three conditions is true :

(1) $3^2 \mid n$,

(2) $P_i \equiv 1 \pmod 3$,

(3) $P_i \equiv 2 \pmod 3$, k is even

where P_i is some prime factor of n.

Theorem 7— $J_k(n) \equiv O \pmod P$ where P is any prime number if one of the following conditions is true.

(1) $P^2 \mid n$

(2) $P_i \equiv 1 \pmod P$, where P_i is some prime factor of n.

PROOF : Let $n = \prod_{i=1}^{\gamma} P_i^{\alpha_i}$ be the canonical representation of n, then

$$J_k(n) = \prod_{i=1}^{\gamma} J_k\left(P_i^{\alpha_i}\right).$$

If $P^2 \mid n$, then $P = P_i$ for some i and correspondingly $\alpha_i \geqslant 2$.

Consequently

$$J_k\left(P_i^{\alpha_i}\right) = P_i^{(\alpha_i-1)k}\left(P_i^k - 1)\right)$$

$$= P^{(\alpha_i-1)k}(P^k - 1) \equiv O \pmod{P}$$

$$(\alpha_i - 1 \geqslant 1).$$

On the other hand, if

$P_i \equiv 1 \pmod{p}$, then $P_i^k \equiv 1 \pmod{p}$,

so that $J_k\left(P_i^{\alpha_i}\right) \equiv O \pmod{p}$.

Hence $J_k(n) \equiv O \pmod{p}$ in either case, establishing the theorem.

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SOME APPLICATIONS OF ARCWISE CONNECTED FUNCTIONS FOR MINIMAX INEQUALITIES AND EQUALITIES

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Some interesting applications of arcwise connected functions, whose properties were discussed by Singh (*J. Optimization Theory Applic.* 41 (1983), 377–87) are given in the area of minimax inequalities and equalities in the present note.

§1. Singh[3] discussed some elementary properties of arcwise connected sets and functions. The purpose of the present note is to study some of the nontrivial applications of arcwise functions in the area of minimax inequalities and equalities. In this context it is to be observed that the arcwise connected property infact is a generalization of the convexity with regard to sets and functions.

§2. *Definition* 2.1—The set $X \subset R^n$ is said to be arcwise connected (AC) if, for every pair of points x^1, x^2 in X, there exists a continuous vector-valued function H_{x^1, x^2}, called an arc, defined on the uuit interval $[0,1] \subset R$ with values in X such that

$$H_{x^1, x^2}(0) = x^1 \text{ and } H_{x^1, x^2}(1) = x^2.$$

For any positive integer k, we let

$$I^k = (a_1, \ldots, a_k) : 0 \leqslant a_i \leqslant 1;$$

i.e., I^k denotes the kth dimensional unit cube. Further let l_i denote the ith unit vector in I^k, i.e.,

$$e_i = (0, \ldots, 0, 1, 0, \ldots, 0);$$

i.e., the ith component of e_i is 1 and all other components are zero.

The following proposition is given in Singh[3].

Proposition 2.1—Suppose $X \subset R^n$, then X is AC if and only if, for any positive integer k and $x^1, x^2, \ldots, x^k$ in X, there exists a continuous function $H_{x^1, x^2, \ldots, x^k}$ defined on I^k such that

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$$H_{x^1, x^2, \ldots, x^k}(e_i) = x^1, \text{ for } i = 1, 2, \ldots, k.$$

Definition 2.2—Let $X \subset R^n$ be an arcwise connected set, and let f be a real-valued function defined on X. We say f is arcwise connected convex (CN) if, for every $x^1, x^2 \in X$, there exists an arc H_{x^1, x^2} in X such that

$$f(H_{x^1, x^2}(\theta)) \leqslant (1-\theta) f(x^1) + \theta f(x^2), \text{ for all } \theta, 0 \leqslant \theta \leqslant 1.$$

We have similar definition for arcwise connected concave function.

In view of Proposition 2.1 we give the following definition for generalized arcwise connected hull of k points $x^1, x^2, \ldots, x^k$.

Definition 2.3—Let $x^1, x^2, \ldots, x^k \in X \subset R^n$. Then the generalized arcwise connected hull (GACH) of $\{x^1, x^2, \ldots, x^k\}$ is the set $\{H_{x^1, x^2, \ldots, x^k}(a_1, a_2, \ldots, a_k) :$ $0 \leqslant a_i \leqslant 1, \sum_{i=1}^{k} a_i = 1\} \subset R^n$ for a unique continuous function defined on I^k.

Now let $X \subset R^n$. Consider the set denoted by $GCo(X)$ as follows.

$$GCo(X) = \{H_{x^1, x^2, \ldots, x^k}(a_1, a_2, \ldots, a_k) : 0 \leqslant a_i \leqslant 1, \sum_{i=1}^{k} a_i = 1,$$
$$x^1, x^2, \ldots, x^k \text{ are finite number of points in } X\}.$$

Remark : (1) In the above definitions of $GCo(X)$ we assume that the function H is unique.

(2) The definition of $GCo(X)$ assume the uniqueness of H in the sense that H does not vary from one tuple $\{x^1, \ldots, x^k\}$ to another tuple $\{x'^1, \ldots, x'^k\}$. For example a function of the following type :

$$H_{x^1, \ldots, x^k} = \sum_{i=1}^{k} a_i x^i$$

would work for an ilustration of the functional form of H as in the case of ordinary convex hull.

Definition 2.4—Let $X \subset R^n$. Let f be a real valued function defined on X. Then f is said to be generalized arcwise connected convex on $GCo(X)$ is given $x^1, x^2, \ldots, x^k \in X$

$$f[H_{x^1, x^2, \ldots, x^k}(a_1, a_2, \ldots, a_k)] \leqslant \sum_{i=1}^{k} a_i f(x^i)$$

with $\sum_{i=1}^{k} a_i = 1$, for all points $H_{x^1, x^2, \ldots, x^k}(a_1, a_2, \ldots, a_k)$ on the GACH of $x^1, x^2, \ldots, x^k$.

We have similar definition for generalized arcwise connected concave function.

§3. As an application of generalized arcwise connected functions we give a minimax inequality derived in Theorem 3.1. We make the following assumptions.

Let $X \subset R^n$ and for each $x \in X$ let a closed set $G(x)$ in R^n be given such that $G(x)$ is compact for at least one $x \in X$. If the GACH of every finite subset $\{x^1, x^2, \ldots, x^k\}$ of X with respect to the same unique are H is contained in the corresponding union $\bigcup_{i=1}^{n} G(x^i)$ then $\bigcap_{x \in X} G(x) \neq \phi$.

The assumption $\bigcap_{x \in X} G(x) \neq \phi$ with the properties satisfied by $\bigcup_{i=1}^{n} G(x^i)$, for each $x_i \in X$, $i = 1, 2, \ldots, n$ and other condition mentioned above is a topological property (which we henceforth denote by G-property) which is verified for the case of ordinary convex hull of $\{x_1, x_2, \ldots, x_n\}$ by Fan's lemmas[1]. The motivation starts at this point when we tag this assumption for the derivation of the minimax inequality given below in Theorem 3.1 for the more general case where X is a compact AC set in R^n with G-property.

Theorem 3.1—Let X be a compact AC set in R^n. Let X also possess G-property. Let f and g be real valued functions on $X \times X$ with the following properties :

(i) For each $x \in X$, $g(x)$ is a lower semi-continuous function on X,

(ii) For each $y \in X$, $f(.\,, y)$ is a generalized arcwise connected concave funtion on X,

(iii) $g(x, y) \leqslant f(x, y)$ for all $(x, y) \in X \times X$, then the minimax inequality

$$\min_{y \in X} \operatorname{Sup}_{x \in X} g(x, y) \leqslant \operatorname{Sup}_{x \in X} f(x, x)$$

holds.

Remark : Theorem 3.1 is in fact a generalization of a result (Theorem 1, p.479) of Yen[2] on R^n.

Proof : Let $t = \sup\{f(x, x) : x \in X\}$. Without loss of generality we may assume that $t < \infty$. For each $x \in X$, let

$$F(x) = \{y \in X : f(x, y) \leqslant t\}$$

$$G(x) = \{y \in X : g(x, y) \leqslant t\}$$

then by (i), (ii) and (iii) we have that, (iv) $G(x)$ is a closed subset of a compact set X and hence $G(x)$ is compact for all $x \in X$.

(v) For any finite set $\{x^1, x^2, \ldots, x^k\}$ we shall show that GACH of $\{x^1, x^2, \ldots, x^k\}$ is a subset of $\bigcup_{i=1}^{k} F(x^i)$. Observe that in view of Proposition 2.1 GACH of

$\{x^1, x^2, \ldots, x^k\}$ is in X. Since $f(. , y)$ is generalized arcwise connected concave for each $y \in X$, we have,

$$\sum_{i=1}^{k} a_i f(x^i, H_{x^1, x^2, \ldots, x^k}(a_1, a_2, \quad , a_k))$$

$$\leqslant f(H_{x^1, x^2, \ldots, x^k}(a_1, a_2, \ldots, a_k),$$

$$H_{x^2, x^1, \ldots, x^k}(a_1, a_2, \ldots, a_k)) \leqslant t. \qquad \ldots(1)$$

From (1) one can see that

$$f(x^i, H_{x^1, x^2, \ldots, x^k}(a_1, a, \ldots, a_k)) \leqslant t$$

for at least one index i, which shows that

$$H_{x^1, x^2, \ldots, x^k}(a_1, a_2, \ldots, a_k) \in \bigcup_{i=1}^{k} F(x^i).$$

(vi) For each $x \in X$, $F(x) \subset G(x)$, respectively because of (iii). Therefore it follows from (v) and (vi) that

(vii) For any finite subset $\{x^1, x^2, \ldots, x^k\}$ of X we have the GACH of $\{x^1, x^2, \ldots, x^k\}$ is a subset of $\bigcup_{i=1}^{k} G(x^i)$. Therefore from the assumption preceeding Theorem 3.1 and the fact that (iv) and (vii) hold, we have that $\cap \{G(x) : x \in X\} \neq \phi$.

Let

$$y_0 \in \{G(x) : x \in X\}.$$

Then

$$g(x, y_0) \leqslant t \text{ for all } x \in X$$

and our minimax inequality holds.

§4. In this section a further application of arcwise connected function is given for minimax equalities on unbounded sets in R^n. The example thus given is inspiredby the work of Hirano and Takahashi[4] for min-max equality for functions $F(. , \ )$ which are convex concave type in respective variables. In what follows $\partial_A K$ and $i_A K$ will denote the boundaries points and interior points of a set K imbedded in a set A in R^n respectively.

Theorem 4.1—Let A, B be two non empty closed arcwise connected (AC) subsets of R^n. If F is a function on $A \times B$ such that for each $y \in B$, $F(. , y)$ is an upper semi-continuous generalized arcwise concave function on A and for each $x \in A$, $F(x , .)$ is a lower semi-continuous generalized arcwise convex function on B, then a sufficient condition for the min-max equality

$$\max_{x \in A} \min_{y \in B} F(x, y) = \min_{y \in B} \max_{x \in A} F(x, y)$$

is given as follows :

There exist bounded closed arcwise convex sets $K \subset A$ and $L \subset B$ such that for each $(x, y) \in (\partial_A K \times L) \cup (K \times \partial_B L)$, there exists $(u, v) \in i_A K \times i_B L$ which satisfies $F(u, y) \geqslant F(x, v)$.

PROOF : Let K and L be two bounded closed arcwise connected subsets satisfying the condition stated in the theorem. Then because of upper semi-continuity and lower semi-continuity conditions on $F(. , y)$ and $F(x , .)$ respectively there exists $(x_0, y_0) \in K \times L$ such that $F(x, y_0) \leqslant F(x_0, y_0) \leqslant F(x_0, y)$ for all $(x, y) \in K \times L$. Let $(x_0, y_0) \in i_A K \times i_B L$. Then for each $x \in A$ we can choose $\theta > 0$ so small that $H_{xx_0}(\theta) \in K$. Since $F(. , y)$ is generalized arcwise concave, we have

$F(x_0, y_0) \geqslant F(H_{xx_0}(\theta), y_0) \geqslant \theta F(x, y_0) + (1 - \theta) F(x_0, y_0)$ and hence $F(x, y_0) \leqslant F(x_0, y_0)$. Also we can get similarly $F(x_0, y_0) \leqslant F(x_0, y)$ for all $y \in B$. Now let $(x_0, y_0) \in (\partial_A K \times L) \cup (K \times \partial_B L)$. Then by the hypothesis in the Theorem there exists $(u, v) \in i_A K \times i_B L$ such that $F(u, y_0) \geqslant F(x_0 v)$ since $F(x, y_0) \leqslant F(x_0, y_0) \leqslant F(x_0, y)$ for all $(x, y) \in K \times L$, we have $F(u, y_0) = F(x_0, y_0) = F(x_0, v)$. For each $x \in A$, we choose $\theta > 0$ so small that $H_{xu}(\theta) \in K$. Then

$$\begin{aligned} F(x_0, y_0) &\geqslant F(H_{xu}(\theta), y_0) \\ &\geqslant \theta F(x, y_0) + (1 - \theta) F(u, y_0) \\ &= \theta F(x, y_0) + (1 - \theta) F(x_0, y_0). \end{aligned}$$

Hence we obtain that $F(x, y_0) \leqslant F(x_0, y_0)$. Also we obtain similarly $F(x_0, y_0) \leqslant F(x_0, y)$ for all $y \in B$, which completes the proof of the theorem.

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NON-CONVEX AND SEMI-DIFFERENTIABLE FUNCTIONS

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This paper defines η-convexity, η-quasiconvexity and η-pseudoconvexity for semi-differentiable functions. Some properties involving these functions are discussed. Sufficient optimality criteria for non-linear programming problems involving these functions are given.

§1. Hanson[2] defined invexity for differentiable functions as a very broad generalization of convexity. A mathematical program of the form :

$$\text{Min } f(x) \text{ subject to } g(x) \leqslant 0,\ x \in D \subseteq R^n$$

is invex if there exists a function $\eta : D \times D \to R^m$ such that for all $x, u \in D$,

$$f(x) - f(u) \geqslant \eta(x, u) \nabla f(u)$$

and

$$g(x) - g(u) \geqslant \eta(x, u) \nabla g(u).$$

It may be noted here that the convex case corresponds to

$$\eta(x, u) = x - u.$$

Kaul and Kaur[6] defined η-convexity, η-quasiconvexity and η-pseudoconvexity for differentiable functions on the similar lines. In this paper, we consider functions which are semi-differentiable i.e. those functions for which the right differential of the functions exist at each point of the set on which the functions are defined.

In section 2, we define the terms η-convexity, η-quasiconvexity, η-pseudoconvexity and other related terms for semi-differentiable functions. In section 3 we mention some of the properties possessed by these functions. Section 4 is devoted to a discussion of the optimality criteria for programming problems involving different generalised invex functions.

§2. We start with recapitulating the Definitions 2.1 and 2.2 from Kaul and Kaur[4].

Let R^n be the n-dimensional Euclidean space and f be a numerical function defined on a set $C \subseteq R^n$.

Definition 2.1— The right differential of f at $\bar{x} \in C$ in the direction of $x - \bar{x}$ denoted by $df^+(\bar{x}, x - \bar{x})$ is defined as

$$df^+(\bar{x}, x - \bar{x}) = \underset{\lambda \to 0^+}{Lt} \frac{f((1 - \lambda)\bar{x} + \lambda x) - f(\bar{x})}{\lambda}$$

provided the limit exists.

If the right differential exists at each $\bar{x} \in C$, then f is said to be semi-differentiable on C.

Definition 2.2— A subset $C \subseteq R^n$ is said to be locally starshaped at $\bar{x} \in C$ if corresponding to $\bar{x}$ and each $x \in C$, there exists a maximum positive number $a(\bar{x}, x) \leqslant 1$ such that

$$(1 - \lambda)\bar{x} + \lambda x \in C, 0 < \lambda < a(\bar{x}, x).$$

If C is locally starshaped for each $\bar{x} \in C$, then C is a locally starshaped set at each of its points.

Definition 2.3— A semi-differentiable numerical function f defined on a set $C \subseteq R^n$ is said to be η-convex at x^* if there exists a numerical function $\eta(x, x^*)$ defined on $C \times C$ such that

$$f(x) - f(x^*) \geqslant \eta(x, x^*)\, df^+(x^*, x - x^*), \forall\, x \in C.$$

f is said to be η-convex on C if there exists a numerical function $\eta(x_1, x_2)$ defined on $C \times C$ such that

$$f(x_1) - f(x_2) \geqslant \eta(x_1, x_2)\, df^+(x_2, x_1 - x_2) \; \forall\, x_1, x_2 \in C. \qquad \ldots(2.1)$$

When the relation (2.1) is satisfied as a strict inequality, f is said to be a strictly η-convex function.

In particular when $\eta(x_1, x_2) = 1$, $\forall\, x_1\, x_2 \in C$ in the inequality (2.1) and the set C is a locally starshaped set, then the function f is said to be semilocally convex.

Remark 2.1 : Every semilocally convex function is η-convex but the converse is not always true.

The following example shows a function which is η-convex but is not semi-locally η-convex.

Example 2.1 — Consider a function

$f : [0, \frac{\pi}{2}[\rightarrow R$ defined by

$$
\begin{aligned}
f(x) &= \sin x, \qquad 0 \leqslant x < \frac{\pi}{6} \\
&= 2 \sin x - \frac{1}{2} \qquad \frac{\pi}{6} \leqslant x < \frac{\pi}{2}.
\end{aligned}
$$

It is clear that the function is not differentiable at $x = \frac{\pi}{6}$. We also have

$$
df^+(x_2, x_1 - x_2) = \begin{cases} (x_1 - x_2)\cos x_2 & \text{for } 0 \leqslant x_1 < \frac{\pi}{6},\ 0 \leqslant x_2 < \frac{\pi}{6} \\ 2(x_1 - x_2)\cos x_2 & \text{for } \frac{\pi}{6} \leqslant x_1 < \frac{\pi}{2},\ \frac{\pi}{6} < x_2 < \frac{\pi}{2} \\ (x_1 - \frac{\pi}{6})\cos\frac{\pi}{6} & \text{for } 0 \leqslant x_1 < \frac{\pi}{6},\ x_2 = \frac{\pi}{6} \\ 2(x_1 - \frac{\pi}{6})\cos\frac{\pi}{6} & \text{for } \frac{\pi}{6} \leqslant x_1 < \frac{\pi}{2},\ x_2 = \frac{\pi}{6}. \end{cases}
$$

Let us choose

$$
\begin{aligned}
\eta(x_1, x_2) &= \frac{\sin x_1 - \sin x_2}{(x_1 - x_2)\cos x_2} \quad \text{when } x_1 \neq x_2 \\
&= 1 \qquad \text{if } x_1 = x_2.
\end{aligned}
$$

Then it can be easily verified that the inequality

$$f(x_1) - f(x_2) \geqslant \eta(x_1, x_2)\, df^+(x_2, x_1 - x_2) \text{ holds in } [0, \frac{\pi}{2}[.$$

Hence the function f is η-convex.

Taking $x_1 = \frac{\pi}{12}$ and $x_2 = \frac{\pi}{18}$, we observe that

$$f(x_1) - f(x_2) \ngeqslant df^+(x_2, x_1 - x_2).$$

This shows that f is not a semilocally convex function.

Definition 2.4— A semi-differentiable function f defined on a set $C \subseteq R^n$ is said to be η-quasiconvex at $x^* \in C$ if there exists a numerical function $\eta(x, x^*)$ defined on $C \times C$ such that

$$f(x) \leqslant f(x^*) \Rightarrow \eta(x, x^*)\, df^+(x^*, x - x^*) \leqslant 0, \ \forall\ x \in C.$$

f is said to be η-quasiconvex on C if there exists a numerical function $\eta\,(x_1, x_2)$ defined on $C \times C$ such that

$$f\,(x_1) \leqslant f\,(x_2) \Rightarrow \eta(x_1, x_2)\, df^{+}\,(x, x_1 - x_2) \leqslant 0, \ \forall\ x_1, x_2 \in C.$$

The function f is called strictly η-quasiconvex when

$$f\,(x_1)\, x_1 \overset{<}{\neq} x_2\, f\,(x_2) \Rightarrow \eta\,(x_1, x_2)\, df^{+}\,(x_2, x_1 - x_2) < 0, \ \forall\ x_1, x_2 \in C.$$

The function f is called strongly η-quasiconvex when

$$f\,(x_1) \underset{x_1 \neq x_2}{\leqslant} f\,(x_2) > \Rightarrow \eta\,(x_1, x_2)\, df^{+}\,(x_2, x_1 - x_2) < 0, \ \forall\ x_1, x_2 \in C.$$

In particular when

$$\eta\,(x_1, x_2) = 1, \ \forall\ x_1, x_2 \in C$$

and C is a locally starshaped set, then we have definition of a semilocally quasiconvex, semilocally explicity quasiconvex and semilocally strongly quasiconvex functions respectively.

Remark 2.3 : Every η-convex function is η-quasiconvex for the same function η but the converse is not true.

From the definition of η-convexity, we find that

$$\eta\,(x_1, x_2)\, df^{+}\,(x_2, x_1 - x_2) \leqslant f\,(x_1) - f\,(x_2), \ \forall\ x_1, x_2 \in C.$$

Therefore

$$f\,(x_1) - f\,(x_2) \leqslant 0 \Rightarrow \eta\,(x_1, x_2)\, df^{+}\,(x_2, x_1 - x_2) \leqslant 0$$

showing that η-convex is also η-quasiconvex for the same function η.

The following example, however, shows that the converse is not always true.

Example 2.2— Consider a function f : $[1, 4[\ \to R$ defined as follows :

$$\begin{aligned} f\,(x) &= x^3, \quad 1 \leqslant x < 2 \\ &= 2x^2, \ 2 \leqslant x < 4. \end{aligned}$$

Clearly this function is not differentiable at $x = 2$. The computation of the right differential of the function yields

$$df^{+}\,(x_2, x_1 - x_2) = \begin{cases} 3\,(x_1 - x_2)\, x_2^2 & \text{for } 1 \leqslant x_1 < 2,\ 1 \leqslant x_2 < 2 \\ 4\,(x_1 - x_2)\, x_2 & \text{for } 2 < x_1 < 4,\ 2 < x_2 < 4 \\ 12\,(x_1 - 2) & \text{for } 1 \leqslant x_1 < 2,\ x_2 = 2 \\ 8\,(x_1 - 2) & \text{for } 2 \leqslant x_1 < 4,\ x_2 = 2. \end{cases}$$

Let us choose $\eta\,(x_1, x_2) = x_1/2$, then it can be easily seen that in all the above ranges of x_1, x_2 we have

$$f\,(x_1) \leqslant f\,(x_2) \Rightarrow \eta\,(x_1, x_2)\, df^{+}\,(x_2, x_1 - x_2) \leqslant 0.$$

Hence the function f is η-quasiconvex. But this function is not η convex, for

$$f(x_1) - f(x_2) \geqslant \eta(x_1, x_2)\, df^+(x_2, x_1 - x_2)$$

does not hold at $x_1 = \frac{5}{4}$, $x_2 = \frac{3}{2}$.

Remark 2.4 : Every semilocally quasiconvex function is η-quasiconvex but the converse is not true, can be seen from the following example.

Example 2.3—Consider a function f defined by

$$f :]-1, 5[\rightarrow R$$

and

$$\begin{aligned} f(x) &= -2x^2, \quad -1 \leqslant x < 2 \\ &= -x^3, \quad 2 \leqslant x < 5. \end{aligned}$$

This function is not differentiable at $x = 2$. The computation of the right differential of the function f yields

$$df^+(x_2, x_1 - x_2) = \begin{cases} -4(x_1 - x_2)\, x_2 & \text{for } -1 \leqslant x_1 < 2, -1 \leqslant x_2 < 2 \\ -3(x_1 - x_2)\, x_2^2 & \text{for } 2 \leqslant x_1 < 5, 2 \leqslant x_2 < 5 \\ -8(x_1 - 2) & \text{for } -1 \leqslant x_1 < 2, x_2 = 2 \\ -12(x_1 - 2) & \text{for } 2 \leqslant x_1 < 5, x_2 = 2. \end{cases}$$

Let us choose

$$\begin{aligned} \eta(x_1, x_2) &= \frac{x_1 + x_2}{x_2} \qquad \text{if } x_2 \neq 0 \\ &= 1 \qquad\qquad\quad \text{if } x_2 = 0, \end{aligned}$$

then it can be easily seen that in all the above ranges of x_1 and x_2, we have

$$f(x_1) \leqslant f(x_2) \Rightarrow \eta(x_1, x_2)\, df(x_2, x_1 - x_2) \leqslant 0.$$

Hence function f is η-quasiconvex.

But this function is not semilocally quasiconvex. This can be easily verified by taking

$$x_1 = -0.5 \text{ and } x_2 = +0.25.$$

We have

$$f(x_2) = -0.125, f(x_1) = -0.5$$

showing that

$$f(x_1) \leqslant f(x_2)$$

and

$$df^+(x_2, x_1 - x_2) = 0.75 \nleqslant 0.$$

Defintion 2.5— A semi-differentiable numerical function f defined on a set $C \subseteq R^n$ is said to be η-pseudoconvex at $x^* \in C$ if there exists a numerical function $\eta(x, x^*)$ defined on $C \times C$ such that

$$\eta(x, x^*)\, df^+(x^*, x - x^*) \geqslant 0 \Rightarrow f(x) > f(x^*),\ \forall\ x \in C.$$

Also f is η-pseudoconvex on C if there exists a numerical function $\eta(x_1, x_2)$ defined on $C \times C$ such that

$$\eta(x_1, x_2)\, df^+(x_2, x_1 - x_2) \geqslant 0 \Rightarrow f(x_1) \geqslant f(x_2),\ \forall\ x_1, x_2 \in C.$$

As a particular case when $\eta(x_1, x_2) = 1$, $\forall\ x_1, x_2 \in C$, the function f is is said to be semilocally pseudoconvex.

Remark 2.5 : Every semilocally pseudoconvex function is η-pseudoconvex but the converse is not true.

Clearly every semilocally pseudoconvex function is η-pseudoconvex where $\eta(x_1, x_2) = 1$, $\forall\ x_1, x_2 \in C$. The following example, however, shows that the converse is not true.

Example 2.4— Consider a function $f : [0, \frac{\pi}{2}[\rightarrow R$ defined by

$$\begin{aligned} f(x) &= \sin x, \quad 0 \leqslant x < \frac{\pi}{6} \\ &= 2\sin^2 x, \quad \frac{\pi}{6} \leqslant x < \frac{5\pi}{6}. \end{aligned}$$

Clearly this function f is not differentiable at $x = \frac{\pi}{6}$. Computing the right dlfferential of the function f, we have

$$df^+(x_2, x_1 - x_2) = \begin{cases} (x_1 - x_2)\cos x_2 \text{ for } 0 \leqslant x_1 < \frac{\pi}{6}, 0 \leqslant x_2 < \frac{\pi}{6} \\ 4(x_1 - x_2)\sin x_2 \cos x_2 \text{ for } \frac{\pi}{6} \leqslant x_1 < \frac{5\pi}{6}, \\ \qquad \frac{\pi}{6} < x_2 < \frac{5\pi}{6} \\ (x_1 - \frac{\pi}{6})\cos\frac{\pi}{6} \text{ for } 0 \leqslant x_1 < \frac{\pi}{6}, x_2 = \frac{\pi}{6} \\ 4(x_1 - \frac{\pi}{6})\sin\frac{\pi}{6}\cos\frac{\pi}{6} \text{ for } \frac{\pi}{6} \leqslant x_1 < \frac{5\pi}{6}, \\ \qquad x_2 = \frac{\pi}{6} \end{cases}$$

Let

$$\begin{aligned} \eta(x_1, x_2) &= \frac{(\sin x_1 - \sin x_2)}{x_1 - x_2}\cos x_2 \text{ if } x_1 \neq x_2 \\ &= 1 \text{ if } x_1 = x_2. \end{aligned}$$

It can be easily shown that in all the above ranges of x_1, x_2 we have

$$\eta(x_1, x_2)\, df^+(x_2, x_1 - x_2) \geqslant 0 \Rightarrow f(x_1) \geqslant f(x_2).$$

Hence the function f is η-pseudoconvex.

But this function f is not semilocally pseudoconvex as

$$df^+(x_2, x_1 - x_2) \geqslant 0 \text{ and } f(x_1) < f(x_2)$$

at

$$x_1 = \frac{\pi}{4} \text{ and } x_2 = \frac{2\pi}{3}.$$

Definition 2.6— An m-dimensional vector function $h = (h_1, \ldots, h_m)$ defined on $C \subseteq R^n$ is η-convex, η-quasiconvex, η-pseudoconvex on C if each of its components h_i $(i = 1, \ldots, m)$ is η-convex, η-quasiconvex, η-pseudoconvex on C respectively.

§3. *Theorem* 3.1— A semidifferentiable numerical function f defined on a set $C \subseteq R^n$ is η-convex iff

$$x_1, x_2 \in C \text{ and } df^+(x_2, x_1 - x_2)\ 0 \Rightarrow f(x_1) \geqslant f(x_2).$$

PROOF: Suppose f is η-convex. Therefore for $x_1, x_2 \in C$,

$$f(x_1) - f(x_2) \geqslant \eta(x_1, x_2)\, df^+(x_2, x_1 - x_2).$$

It follows from this inequality that

$$df^+(x_2, x_1 - x_2) = 0 \Rightarrow f(x_1) \geqslant f(x_2).$$

Conversely let $f(x_1) \geqslant f(x_2)$ for $x_1, x_2 \in C$,

whenever $df^+(x_2, x_1 - x_2) = 0$. We need to show that

$$f(x_1) - f(x_2) \geqslant \eta(x_1, x_2)\, df^+(x_2, x_1 - x_2), \forall\, x_1, x_2 \in C. \qquad \ldots(3.1)$$

If $df^+(x_2, x_1 - x_2) = 0$, then the inequality (3.1) holds in view of the given hypothesis.

If $df^+(x_2, x_1 - x_2) \neq 0$ then choose

$$\eta(x_1, x_2) = \frac{f(x_1) - f(x_2)}{df^+(x_2, x_1 - x_2)}$$

and inequality (3.1) is again verified.

Hence f is η-convex.

This completes the proof of the theorem.

Theorem 3.2— Let f be a numerical function defined on the set $C \subseteq R^n$ and let f be semi-differentiable at $x^* \in C$. Suppose there exists a positive numerical func-

tion $\eta(x, x^*)$ defined on $C \times C$ and maximum positive numbers $a(x,x^*)$ and $d(x,x^*)$ such that

$$x^* + \lambda \eta(x, x^*)(x - x^*) \in C, 0 < \lambda < a(x, x^*)$$

and

$$f(x^* + \lambda \eta(x, x^*)(x - x^*)) \leqslant (1 - \lambda) f(x^*) + \lambda f(x),$$
$$\forall x \in C, 0 < \lambda < d(x, x^*),$$

where

$$a(x, x^*) \leqslant 1 \text{ and } d(x, x^*) \leqq a(x, x^*).$$

Then f is η-convex at x^*.

PROOF : We have

$$f(x^* + \lambda \eta(x, x^*)(x - x^*)) \leqslant (1 - \lambda) f(x^*) + \lambda f(x)$$
$$\forall x \in C, 0 < \lambda < d(x, x^*).$$

Therefore,

$$\frac{[f(x^* + \lambda \eta(x, x^*))(x - x^*) - f(x^*)]}{\lambda \eta(x, x^*)} \times \eta(x, x^*) \leqslant f(x) - f(x^*)$$
$$\forall x \in C, 0 < \lambda < d(x, x^*).$$

Taking limit as $\lambda \to 0^+$, immediately yields the inequality

$$df^+(x^*, x - x^*) \eta(x, x^*) \leqslant f(x) - f(x^*), \forall x \in C.$$

Hence f is η-convex at x^*.

Assuming $\eta(x, x^*) = 1$, $\forall x \in C$ and C a locally starshaped set, we obtain a particular case of the above result regarding a semilocally convex function.

Theorem 3.3— Let a numerical function f defined on a set $C \subseteq R^n$ be semi-differentiable at x^*. Suppose there exists a positive numerical function $\eta(x, x^*)$ defined on $C \times C$, maximum positive numbers $a(x, x^*)$ and $d(x, x^*)$ such that

$$a(x, x^*) < 1, d(x, x^*) < a(x, x^*),$$

$$x^* + \lambda \eta(x, x^*)(x - x^*) \in C, 0 < \lambda < a(x, x^*)$$

and

$$f(x) \leqslant f(x^*) \Rightarrow f(x^* + \lambda \eta(x, x^*)(x - x^*)) \leqslant f(x^*),$$
$$\forall x \in C, 0 < \lambda < d(x, x^*)$$

then f is η-quasiconvex at x^*.

PROOF : It is given that

$$f(x) \leqslant f(x^*) \Rightarrow f(x^* + \lambda \eta(x, x^*)(x - x^*)) \leqslant f(x^*),$$
$$\forall x \in C, 0 < \lambda < d(x, x^*).$$

Therefore

$$f[x^* + \lambda\, \eta(x, x^*)(x - x^*)] - f(x^*) \leqslant 0,$$
$$\forall\, x \in C,\, 0 < \lambda < d(x, x^*)$$

i.e.

$$\frac{[f(x^* + \lambda\, \eta(x, x^*)(x - x^*)) - f(x^*)]}{\lambda\, \eta(x, x^*)}\, \eta(x, x^*) \leqslant 0,$$
$$\forall\, x \in C,\, 0 < \lambda < d(x, x^*).$$

Taking limit as $\lambda \to 0^+$, we have

$$\eta(x, x^*)\, df^+(x^*, x - x^*) \leqslant 0,\ \forall\, x \in C.$$

Hence the function f is η-quasiconvex at x^*.

As a particular case when $\eta(x, x^*) = 1$, $\forall\, x \in C$ and C is a locally starshaped set then Theorem 3.3 expresses results for a semi-locally quasiconvex function.

§4. *Sufficient Optimality Criterion*— Consider the non-linear programming problem (P) : Min. $f(x)$

subject to $g(x) \leqslant 0$

$x \in C$

where f and g are semi-differentiable numerical and m-dimensional vector function respectively defined on a set $C \subseteq R^n$.

Let $X = \{x \in C, g(x) \leqslant 0\}$ be the set of all feasible solution of (P).

Theorem 4.1— Let $x^* \in C$ and let f and g be η-convex at x^* for the same function η. If there exist $u_0^* \in R$ and $u^* \in R^m$ such that (x^*, u_0^*, u^*) satisfy the following conditions

$$u_0^*\, \eta(x, x^*)\, df^+(x^*, x - x^*) + \eta(x, x^*)\, u^*\, dg^+(x^*, x - x^*) \geqslant 0,$$
$$\forall\, x \in X \qquad \text{...(4.1)}$$

$$g(x^*) \leqslant 0 \qquad \text{...(4.2)}$$

$$u^{*\prime} g(x^*) = 0 \qquad \text{...(4.3)}$$

$$(u_0^*, u^{**}) \geqslant 0 \qquad \text{...(4.4)}$$

$$u_0^* > 0 \qquad \text{...(4.5)}$$

then x^* is an optimal solution of (P).

PROOF : Since f is η-convex at x^*, therefore for any $x \in X$,

$$f(x) - f(x^*) \geqslant \eta(x, x^*)\, df^+(x^*, x - x^*)$$

$$\geqslant \frac{-\eta(x, x^*)\, u^*\, dg^+(x^*, x - x^*)}{u_0^*}$$

$$\geqslant \frac{u^{*\prime}}{u_0^*}[g(x^*) - g(x)]$$

$$= \frac{-u^{*\prime}}{u_0^*} g(x) \qquad \ldots(4.6)$$

where the second inequality follows from (4.1) and (4.5), the third inequality follows from the η-convexity of g at x^* and the fourth inequality on using (4.2). Making use of (4.4) and the fact that $x \in X$ in (4.6) yields the inequality

$$f(x) \geqslant f(x^*). \qquad \ldots(4.7)$$

(4.2) shows that x^* is feasible for the problem (P), therefore it follows from (4.7) that x^* is indeed optimal.

Corollary 4.1— Let $x^* \in C$ and let f and g be η-convex at x^* for the same function η. If there exists $u^* \in R^m$ such that (x^*, u^*) satisfy the following conditions

$$\eta(x, x^*)\, df^+(x^*, x - x^*) + \eta(x, x^*)\, u^*\, dg^+(x^*, x - x^*) \geqslant 0, \quad \forall\, x \in X. \qquad \ldots(4.8)$$

$$g(x^*) \leqslant 0 \qquad \ldots(4.9)$$

$$u^{*\prime} g(x^*) = 0 \qquad \ldots(4.10)$$

$$u^* \geqslant 0 \qquad \ldots(4.11)$$

then x^* is an optimal solution of (P).

Remark 4.1 : In the Theorem 4.1, since $u^* \geqslant 0$ $g(x^*) \leqslant 0$ and $u^{*\prime} g(x^*) = 0$, we have

$$u_i^* g_i(x^*) = 0, \; i = 1, \ldots, m. \qquad \ldots(4.12)$$

If

$$I = \{i \mid g_i(x^*) = 0\} \text{ and } J = \{i \mid g_i(x^*) < 0\}$$

then

$$I \cup J = \{1, 2, \ldots, m\}.$$

It now follows from (4.12) that $u_i^* = 0$ for $i \in J$. In fact η-convexity of g_I at x^* is all that is needed and not η-convexity of g.

Theorem 4.2— Let $x^* \in C$ and let f be η-convex at x^* and g_I be strictly η-convex at x^* for the same function η. If there exists $u_0^* \in R$ and $u^* \in R^m$, such that (x^*, u_0^*, u^*) satisfy conditions (4.1) — (4.5) of Theorem 4.1 then x^* is an optimal solution of (P) where I and J are defined as above in Remark 4.1.

PROOF: (4.2) — (4.4) give $u_i^* \, g_i(x^*) = 0$, $i = 1, \ldots, m$, therefore

$$u_i^* = 0 \text{ for } i \in J.$$

Now the conditions (4.1) and (4.4) of Theorem 4.1 can be rewritten as

$$u_0^* \, \eta(x, x^*) \, df^+(x^*, x - x^*) + \sum_{i \in I} u_i^* \, \eta(x, x^*) \, dg_i^+(x^*, x - x^*) \geqslant 0, \quad \forall\, x \in X$$

$$(u_0^*, u_I^*) \geqslant 0, \; u_0^* > 0$$

which shows that the system

$$\eta(x, x^*) \, df^+(x^*, x - x^*) < 0$$

$$\eta(x, x^*) \, dg_I^+(x^*, x - x^*) < 0 \qquad \ldots(4.13)$$

has no solution $x \in X$.

We assert that x^* is an optimal solution of (P)

i.e. $f(x) \geqslant f(x^*)$, $\forall\, x \in X$.

Let if possible there exists $x^0 \in X$ such that

$$f(x^0) < f(x^*) \qquad \ldots(4.14)$$

since

$$x^0 \in X \text{ and } g_I(x^0) \leqslant 0 = g_I(x^*) \qquad \ldots(4.15)$$

and f is η-convex and g_I is strictly η-convex at x^* for the same function η, therefore

$$0 > f(x^0) - f(x^*) \geqslant \eta(x, x^*) \, df^+(x^*, x^0 - x^*) \qquad \ldots(4.16)$$

$$0 \geqslant g_I(x^0) - g_I(x^*) > \eta(x, x^*) \, dg_I^+(x^*, x^0 - x^*). \qquad \ldots(4.17)$$

Now (4.16) and (4.17) show that x^0 is a solution of the system (4.13), which gives a contradiction.

Hence x^* is an optimal solution of (P).

Theorem 4.3—Let $x^* \in C$ and let I and J be defined as in Remark 4.1. Let f be η-pseudoconvex at x^* and g_I be η-quasiconvex at x^* for the same function η. If there exists $u^* \in R^m$ such that (x^*, u^*) satisfy conditions (4.8) — (4.11) of Cor. 4.1, then x^* is an optimal solution of (P).

PROOF : It is easy to see that $u_i^* = 0$ for $i \in J$ i.e.

$$u_J^* = 0. \qquad \text{...(4.18)}$$

The function g_I is η-quasiconvex at x^*.
Therefore

$$g_I(x) \leqslant 0 = g_I(x^*), \ \forall\ x \in X$$

implies

$$\eta(x, x^*)\ dg_I^+(x^*, x - x^*) \leqslant 0, \ \forall\ x \in X. \qquad \text{...(4.19)}$$

From (4.11) and (4.19) we obtain

$$\eta(x, x^*)\ u_I^*\ dg_I^+(x^*, x - x^*) \leqslant 0, \ \forall\ x \in X. \qquad \text{...(4.20)}$$

Using (4.18) and (4.19) in (4.8) we have

$$\eta(x, x^*)\ df^+(x^*, x - x^*) \geqslant 0, \ \forall\ x \in X.$$

Since f is η-pseudoconvex at x^*, therefore

$$f(x) \geqslant f(x^*), \ \forall\ x \in X.$$

Hence x^* is an optimal solution of (P).

Theorem : 4.4— Let $x^* \in C$ and $u^* \in R^m$ be such that (x^*, u^*) satisfy conditions (4.8) — (411) of Cor. 4.1. Suppose f is η-pseudoconvex at x^* and $u_I^*\ g_I$ is η-quasiconvex at x^* for the same function η, then x^* is an optimal solution of (P).

PROOF : The proof is same as that of Theorem 4.3 except that we get the relation (4.19) as follows

$$u_I^*\ g_I(x) \leqslant 0 = u_i^*\ g_I(x^*), \ \forall\ x \in X,$$

and $u_I^*\ g_I$ is η-quasiconvex at x^*, therefore

$$\eta(x, x^*)\ u_I^*\ dg_I^+(x^*, x - x^*) \leqslant 0, \ \forall\ x \in X.$$

Theorem 4.5— Let $x^* \in C$ and $u^* \in R^m$ satisfy conditions (4.8) – (4.11) of Corollary 4.1. Let the numerical function $f + u_I^* g_I$ be η-pseudoconvex at x^* for same η, then x^* is an optimal solution of (P).

PROOF : As $u_J^* = 0$, therefore (4.8) can be written as

$$\eta(x, x^*)\, df^+(x^*, x - x^*) + \eta(x, x^*)\, u_I^*\, dg_I^+(x^*, x - x^*) > 0,$$
$$\forall\, x \in X,$$

i.e.

$$\eta(x, x^*)\,(df^+ + u_I^*\, dg_I^+)(x^*, x - x^*) \geqslant 0,\ \forall\, x \in X$$

since $f + u_I^* g_I$ is η-pseudoconvex at x^*, therefore

$$f(x) + u_I^*\, g_I(x) \geqslant f(x^*) + U_I^*\, g_I(x^*),\ \forall\, x \in X.$$

By definition of I and (4.11) we get

$$f(x) \geqslant f(x^*),\ \forall \in X.$$

Hence x^* is an optimal solution of (P).

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ON HYPERCONNECTED SPACES

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A topological space is hyperconnected if intersection of any two non-empty open sets is non-empty. This paper gives a characterisation of hyperconnected spaces, using the concept of semi-open sets, which yields an alternate proof of Noiri's result[8] that hyperconnectedness is a semi-topological property. Further it is proved that a hyperconnected door space is maximal hyperconnected and minimal door and analyse certain related concepts too.

Introduction

Levine[6] called a topological space X a D-space if every nonempty open sub set of X is dense in X. Pipitone and Russo[9] defined a topological space to be semi-connected if it is not the union of two non-empty disjoint semi-open sets and showed that a topological space is semi-connected if and only if it is a D-space. Maheshwari and Tapi[7] defined a topological space X to be S-connected if X is not the union of two non-empty semi-separated sets and showed the equivalence of semi-connectedness and S-connectedness. Sharma[11] indicated that a space is a D-space if it is a hyperconnected spcce due to Steen and Seeback[10]. On the other hand, Strecker[12] has proved that in a topological space 'every non-empty collection of non-empty open sets form a filter base if and only if it is totally co-indiscrete' and the notion of 'irreducible' due to Serré[13] and that of 'superconnected' due to De Groot[4] are shown equivalent to it. De Groot[4] proved that any metrizable or locally compact, Hausdorff space which is not compact has a dual compact, superconnected space which completely determing it.

Preliminaries

Levine[5] defined a subset A of topological space X, semi-open, if there exists an open set U in X, such that, $U \subset A \subset \bar{U}$, where $(-)$ denotes closure in X. We denote the collection of all semi-open sets in a topological space (X, τ) by $SO\,(X, \tau)$. Note that in a hyperconnected space, a non-empty set is semi-open, if and only if, it contains a non-empty open set.

Hyperconnected as a Semi-topological Property

Theorem 1—A topological space (X, τ) is hyperconnected, if and only if $SO\,(X,\tau) \setminus \{\varphi\}$ is a filter on X.

PROOF : Let (X, τ) be hyperconnected. If $A, B \in SO(X, \tau) \setminus \{\varphi\}$, then there exists $U, V \in \tau \setminus \{\varphi\}$ such that $U \subset A$ and $V \subset B$. Since (X, τ) is hyperconnected $\varphi \neq U \cap V \subset A \cap B$ and hence $A \cap B \in SO(X, \tau) \setminus \{\varphi\}$. Now let $A \in SO(X, \tau) \setminus \{\varphi\}$ and $B \supset A$. Then there exists $U \in \tau \setminus \{\varphi\}$ such that $U \subset A \subset B$ and thus $B \in SO(X, \tau) \setminus \{\varphi\}$. Hence $SO(X, \tau) \setminus \{\varphi\}$ is a filter on X. Since $\tau \subset SO(X, \tau)$, sufficiency part is obvious.

Remark : The equivalence class of all topologies on a set X, which have the same semi-open sets as τ, is denoted by $[\tau]$. In Crossley and Hildebrand[2], it is established that $[\tau]$ is a subsemilattice of the lattice of all topologies on X with a greatest element, denoted as $F(\tau)$, with respect to the usual joint operation on topologies.

Theorem 2—Let (X, τ) be a hyperconnected space. Then (X, S), where $S \in [\tau]$ is also hyperconnected. Moreover, $F(\tau) = SO(X, \tau)$.

PROOF : Since $S \in [\tau]$, $SO(X, S) = SO(X, \tau)$ and from Theorem 1, it follows that (X, S) is hyperconnected.

Since $SO(X, \tau) \setminus \{\varphi\}$ is a filter on X, by Theorem 1, $(X, SO(X, \tau))$ is a hyperconnected topological space. If A is semi-open in $(X, SO(X, \tau))$, then there exists $V \in SO(X, \tau) \setminus \{\varphi\}$ such that $V \subset A$. Now V contains a non-empty open set in (X, τ) and hence A is semi-open in (X, τ). Since all the semi-open sets in (X, τ) are clearly semi-open in $(X, SO(X, \tau))$, $SO(X, \tau) \in [\tau]$. But in general, $F(\tau) \subset SO(X, \tau)$. Thus $F(\tau) = SO(X, \tau)$.

Definition—A topological property R is called contractive (expansive) if (X, τ) has the property R and $\tau' \subset \tau$ $(\tau' \supset \tau)$, then (X, τ') has property R.

Remark : Hyperconnectedness is a contractive topological property.

Definition—A topological property preserved under semi-homeomorphisms, which are bijections so that images of semi-open sets are semi-open and inverse images of semi-open sets are also semi-open is called a semi-topological property.

Remark : Regularity, complete regularity, normality, T_3, T_4, T_5, metrizability are known to be not semi-topological; whereas, T_2, first category, separable are semi-topological properties. Noiri[8] has shown that hyperconnectedness is a semi-topological property. We obtain this result as a corollary to Theorem 2 and previous remark.

Let $f : (X, \tau) \to (Y, S)$ be a semi-homeomorphism and (X, τ) be hyperconnected. Then by Theorem 2, $(X, F(\tau))$ is hyperconnected. Since $f : (X, F(\tau)) \to (Y, F(S))$ is a homeomorphism[3], $(Y, F(S))$ is hyperconnected. But $S \subset F(S)$ and hence by previous remark, (Y, S) is hyperconnected.

2. MAXIMAL HYPERCONNECTED SPACES

In this section we analyse maximal hyperconnected spaces and characterise hyperconnected door topologies on a set. Further it is established that a hyper-

connected space is maximal hyperconnected if and only if it is sub-maximal.

Theorem 3—If a topological space (X, τ) is maximal hyperconnected then $SO(X, \tau)\setminus\{\varphi\}$ is an ultrafilter on X and $\tau = SO(X, \tau)$.

PROOF : By Theorem 1, $SO(X, \tau)\setminus\{\varphi\}$ is a filter. Let $A \cup X$, such that $A \notin SO(X, \tau)\setminus\{\varphi\}$. Then $A \notin \tau$. Consider $\tau(A)$, the simple expansion of τ by A. Since $\tau \underset{\neq}{\subset} \tau(A)$, $\tau(A)$ is not hyperconnected. Then there exists two non-empty disjoint open sets, say, C_1 and C_2 in $(X, \tau(A))$. Let $C_1 = U_1 \subset (V_1 \cap A)$ and $C_2 = U_2 \cup (V_2 \cap A)$, where $U_1, U_2, V_1, V_2 \in \tau$.

Now $C_1 \cap C_2 = \varphi \Rightarrow U_1 \cap U_2 = \varphi$; $U_1 \cap V_2 \cap A = \varphi$; $U_2 \cap V_1 \cap A = \varphi$ and $V_1 \cap V_2 \cap A = \varphi$. Since (X, τ) is hyperconnected, $U_1 \cap U_2 = \varphi \Rightarrow U_1 = \varphi$ or $U_2 = \varphi$. We assume, $U_1 = \varphi$.

Two cases may arise.

Case 1 : $U_2 = \varphi$.

Then $V_1 \neq \varphi$ and $V_2 \neq \varphi$, otherwise $C_1 = \varphi$ or $C_2 = \varphi$. Thus we have, $V_1 \cap V_2 \neq \varphi$. Now $V_1 \cap V_2 \cap A = \varphi \Rightarrow \varphi \neq V_1 \cap V_2 \subset A^c \Rightarrow A^c \in SO(X, \tau)\setminus\{\varphi\}$.

Case 2 : $U_2 \neq \varphi$.

Since $C_1 \neq \varphi$, we have $V_1 \neq \varphi$. Then, $U_2 \cap V_1 \neq \varphi$. But $U_2 \cap V_1 \cap A = \varphi$ and hence $A^c \in SO(X, \tau)\setminus\{\varphi\}$. Thus, in either case $SO(X, \tau)\setminus\{\varphi\}$ is an ultrafilter.

By Theorem 2, $(X, SO(X, \tau))$ is hyperconnected and, in general, $\tau \subset SO(X, \tau)$. Since (X, τ) is maximal hyperconnected, $\tau = SO(X, \tau)$.

Theorem 4—Let (X, τ) be a topological space such that $SO(X, \tau)\setminus\{\varphi\}$ is an ultrafilter. Then $(X, SO(X, \tau))$ is maximal hyperconnected.

PROOF : $(X, SO(X, \tau))$ is hyperconnected, obvious. Suppose it is not maximal hyperconnected. Then there exists a hyperconnected space (X, τ_1) such that $SO(X, \tau) \underset{\neq}{\subset} \tau_1$. But then $SO(X, \tau) \underset{\neq}{\subset} SO(X, \tau_1)$ which leads to a contradiction, since by Theorem 1, $SO(X, \tau_1)\setminus\{\varphi\}$ is a filter. Hence the result.

Definition : A topological space X is a door space if for each subset A of X, either A or A^c is open.

Remark : The property of being a door space is an expansive topological property. Steiner[14] characterised door topologies lattice theoretically and it follows that minimal door topologies on a set X are precisely of the form $\{G \subset X \mid x \notin G\} \cup \{X\}$, for some $x \in X$ and $\{\varphi\} \cup \mathscr{U}$, where $\mathscr{U}$ is an ultrafilter on X.

Theorem 5—(X, τ) is a hyperconnected door space if and only if $\tau\setminus\{\varphi\}$ is an ultrafilter on X.

PROOF : *Necessity*—If $A, B \in \tau \setminus \{\varphi\}$, then clearly $A \cap B \in \tau \setminus \{\varphi\}$. Let $A \in \tau \setminus \{\varphi\}$ and $B \supset A$. If $B = X$, then $B \in \tau \setminus \{\varphi\}$. Now assume $B \neq X$. Then $B \in \tau \setminus \{\varphi\}$; otherwise $B^c \in \tau \setminus \{\varphi\}$, since (X,τ) is a door space and then $A \cap B^c = \varphi$, a contradiction. Thus $\tau \setminus \{\varphi\}$ is an ultrafilter.

Sufficiency—Obvious by the previous remark.

Theorem 6—Any hyperconnected door space is maximal hyperconnected and minimal door.

PROOF : Let (X, τ) be a hyperconnected door space. By Theorem 5, $\tau \setminus \{\varphi\}$ is an ultrafilter and hence by the remark (X, τ) is minimal door. Since $\tau \subset SO(X, \tau)$, in general, in view of Theorem 1, $\tau = SO(X, \tau)$, and then by Theorem 4, (X, τ) is maximal hyperconnected.

Remark : Any maximal hyperconnected space is minimal door, but there are minimal door spaces which are not hyperconnected. Let $X = \{a, b, c\}$ and $\tau = \{\varphi, \{a\}, \{b\}, \{a, b\}, X\}$. Then (X, τ) is minimal door, but not even connected.

Definition—A topological space is called submaximal if every dense subset is open.

Theorem 7—Every hyperconnected submaximal space is maximal hyperconnected and conversely.

PROOF : Let (X, τ) be hyperconnected and submaximal.

Now assume (X, τ_1) is hyperconnected such that $\tau_1 \supset \tau$. Let $\varphi \neq 0 \in \tau_1$. Then $\bar{0}_{\tau_1} = X \Rightarrow \bar{0}_{\tau} = X \Rightarrow 0 \in \tau$.

Thus $\tau_1 = \tau$, i.e. (X, τ_1) is maximal hyperconnected. Conversely, let (X, τ) be maximal hyperconnected. Suppose $A \underset{\neq}{\subset} X$ is dense in (X, τ). By Theorem 3, $\tau \setminus \{\varphi\}$ is an ultrafilter and hence A is open. Thus (X, τ) is submaximal.

Remark : Though every maximal connected space is submaximal, a connected submaximal space need not be maximal connected[15].

3. DOWNWARD DIRECTED TOPOLOGICAL SPACES

Each topology τ on a set X may be associated with a pre-order $\rho(\tau)$ on X defined by $(a, b) \in \rho(\tau)$ if and only if every open set containing b contains a. Although the correspondence is many-one, there is always a least topology $\mu(R)$ and a greatest topology $\nu(R)$, having a given pre-order R. Andima and Thron[1] defined a topological space (X, τ) downward directed if and only if each pair of elements in $(X, \rho(\tau))$ has a lower bound. We analyse the relation between the concepts of downward directedness and hyperconnectedness in this section.

Theorem 8—Any downward directed topological space is hyperconnected.

PROOF : Let A and B be non-empty open sets in a downward directed space (X, τ). Let $x \in A$ and $y \in B$. Then there exists $z \in X$ such that $z \rho(\tau) x$ and $z \rho(\tau) y$. Now, $z \in A$ and $z \in B$ and hence $A \cap B \neq \varphi$. ie (X, τ) is hyperconnected.

Remark : Let X be an infinite set with cofinite topology C. Then (X, C) is hyperconnected but not downward directed since the induced order $\rho(C)$ is the diagonal in $x \times X$.

A topological space in which arbitrary intersections of open sets are open is called a principal space. In Andima and Thron[1] it is shown that if (X, τ) is a principal space, then $\tau = \nu(\rho(\tau))$, where $\nu(R)$ for a pre-order R on X is the topology generated by $\{\hat{\{x\}} \mid x \in X\}$ so that $\hat{\{x\}} = \{y \in X \mid y R x\}$.

Theorem 9—Any principal hyperconnected space is downward directed.

PROOF : Let $x, y \in (X, \tau)$, a principal hyperconnected space. Then $\hat{\{x\}}$ and $\hat{\{y\}}$ are non-empty and open in (X, τ) and hence $\hat{\{x\}} \cap \hat{\{y\}} \neq \varphi$. Choose $z \in \hat{\{x\}} \in \hat{\{y\}}$. Then $z \rho(\tau) x$ and $z \rho(\tau) y$ i.e. z is a lower bound of x and y in $(X, \rho(\tau))$. Hence (X, τ) is downward directed.

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SUBMERSIONS OF *CR*-SUBMANIFOLDS OF A KAEHLER MANIFOLD

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For the submersion $\pi : M \to B$ of a CR-submanifolds of a Kaehler manifold $\bar{M}$ onto an almost hermitian manifold B, Kobayashi proved that B becomes a Kaehler manifold. The object of the present paper is to study the impact of the submersion on the Geometry of CR-submanifold M as well as to obtain conditions under which $\bar{M}$ and B are holomorphically isocurved. We have also obtained the Ricci curvatures as well as scalar curvatures of the manifolds $\bar{M}$ and B.

1. INTRODUCTION

The study of submersion $\pi : M \to B$ of a Riemannian manifold M onto Riemannian manifold B was initiated by O'Neill[7,8]. A submersion naturally gives rise to two distributions on M called the horizontal and vertical distribution of which the vertical distribution is always integrable giving rise to fibres which are closed submanifolds of M. Also on a CR-submanifold M of a Kaehler manifold $\bar{M}$ with almost complex structure J there are two natural distributions D and $D^{\perp}$, D being invariant under J and $D^{\perp}$being totally real as well as always integrable[1,2,5]. Kobayashi observed this similarity between the total space of the submersion $\pi : M \to B$ of a CR-submanifold M of a Kaehler manifold $\bar{M}$ onto an almost hermitian manifold B such that the distributions D, $D^{\perp}$ of M become respectively the horizontal and vertical distributions required by the submersion and π restricted to D becomes a complex isometry[6]. He has proved that B in such situation becomes a Kaehler manifold and obtained a relation between the holomorphic sectional curvatures of $\bar{M}$ restricted to D and B. With this naturally following questions arise :

(i) What is the impact of the submersion $\pi : M \to B$ on thc geometry of CR-submanifold M?

(ii) If $\bar{M}$ is a complex space form, under what conditions B is a complex space form?

The object of this paper is to answer these questions (cf sections 3 and 4) as well as we obtain same relations between the Ricci curvatures and the scalar curvatures of the Kaehler manifold and the base manifold.

2. Preliminaries

Let $\bar{M}$ be a Kaehler manifold of real dimension $2n$ with almost complex structure J and hermitian metric g. Then on $\bar{M}$ we have

$$\bar{\nabla}_X JY = J\bar{\nabla}_X Y,\ X, Y \in \mathscr{X}(\bar{M}). \qquad \text{...(2.1)}$$

$\bar{\nabla}$ being the Riemannian connection on $\bar{M}$ and $\mathscr{X}(\bar{M})$ is the lie-algebra of vector fields on $\bar{M}$. An m-dimensional submanifold M of $\bar{M}$ is said to be a CR-submanifold if on M there exist two distributions D and $D^{\perp}$ satisfying $JD = D$ and $JD^{\perp} \subset \nu$, ν being the normal bundle of M (cf. Benjancu[1]). In what follows we shall always take $JD^{\perp} = \nu$, so that if dim $D = 2p$, dim $D^{\perp} = q$ then $m = 2(p+q)$. The Riemannian connection $\bar{\nabla}$ induces Riemannian connections ∇ and $\nabla^{\perp}$ on M and in the normal bundle ν respectively satisfying

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \qquad \text{...(2.2)}$$

$$\bar{\nabla}_X N = -\tilde{A}_N X + \nabla^{\perp}_X N,\quad X, Y \in \mathscr{X}(M),\ N \in \nu \qquad \text{...(2.3)}$$

where h and $\tilde{A}_N$ are the second fundamental form and the Weingarten map respectively, satisfying $g(h(X, Y), N) = g(A_N X, Y)$. It is known that the distribution D is integrable if and only if[1]

$$h(X, JY) = h(X, YJ),\ \forall\ X, Y \in D. \qquad \text{...(2.4)}$$

If $h \equiv 0$, then M is said to be totally geodesic and if $h(X, Y) = g(X, Y)H$, where $H = 1/m$ (trace$h$), then M is said to be totally umbilical CR-submanifold of $\bar{M}$. Let $\bar{R}$, R and $R^{\perp}$ be the curvature tensors corresponding to the connections $\bar{\nabla}$, ∇ and $\nabla^{\perp}$ respectively. Then the equations of Gauss, Codazzi and Ricci are

$$\begin{aligned}\bar{R}(X, Y, Z, W) = R(X, Y, Z, W) &- g(h(X, W), h(Y, Z))\\ &+ g(h(X, Z), h(Y, W))\end{aligned} \qquad \text{...(2.5)}$$

$$[\bar{R}(X, Y)\ Z]^{\perp} = (\bar{\nabla}_X h)(Y, Z) - g(\bar{\nabla}_Y h)(X, Z) \qquad \text{...(2.6)}$$

$$\bar{R}(X, Y, N, N') = R^{\perp}(X, Y; N, N') - g([\tilde{A}_N, \tilde{A}_{N'}](X), Y) \qquad \text{...(2.7)}$$

for X, Y, Z, W tangent to M and $N, N' \in \nu$,

where $[\quad]^{\perp}$ denotes the normal component, and

$$(\bar{\nabla}_X h)(Y, Z) = \nabla^{\perp}_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

For the theory of submersion we follow O'Neill[7]. Let B be an almost Hermitian manifold and we assume that there is a submersion $\pi : M \to B$ of CR-submanifold M onto B such that[6]

(i) $D^\perp$ is Kernel of π_*, that is, $\pi_* D^\perp = \{0\}$

and

(ii) $\pi_* : D_p \to D^*_{\pi(p)}$ is complex isometry, $p \in M$, where $D^*_{\pi(p)}$ denotes the tangent space of B at $\pi(p)$.

(iii) J interchanges $D^\perp$ and ν.

A vector field X on M is said to be basic if

(i) $X \in D$

and

(iii) X is π-related to a vector field on B, i. e., there exists a vector field X_* on B such that $(\pi_* X)_p = X_{*\pi(p)}$ for every $p \in M$.

We have the following lemma for basic vector fields:

Lemma 2.1[7]—Let X and Y be basic vector fields on M. Then

(i) $g(X, Y) = g_*(X_*, Y_*)\, 0\pi$, g_* being Hermitian metric on B

(ii) The horizontal part $P[X, Y]$ of $[X, Y]$ is a basic vector field and corresponds to $[X_*, Y_*]$, i. e. $\pi_*[X, Y] = [X_*, Y_*]$.

(iii) $[V, X] \in D$, $V \in D^\perp$

(iv) $P(\nabla_X Y)$ is basic vector field corresponding to $\nabla^*_{X_*} Y_*$, where ∇^* is Riemannian connection on B.

Put

$$\bar{\nabla}^*_X Y = P(\nabla_X Y), \; X, Y \in D,$$

then $\bar{\nabla}^*_X Y$ is basic vector field and we have

$$\pi_*(\bar{\nabla}^*_X Y) = \nabla^*_{X_*} Y_*. \qquad \ldots(2.8)$$

Define a tensor field C by

$$\nabla_X Y = \tilde{\nabla}^*_X Y + C(X, Y), \; Q(\nabla_X Y) = C(X, Y). \qquad \ldots(2.9)$$

It has been observed in Kobayashi[6] that C is skew-symmetric and we have

Lemma 2.2[6]—If $X, Y \in D$, then

$$C(X, Y) = \frac{1}{2} Q[X, Y].$$

For $X \in D$ and $V \in D^{\perp}$, define A by

$$\nabla_X V = Q(\nabla_X V) + A_X V$$

where $Q(\nabla_X V)$ denotes the vertical part of $\nabla_X V$. Since $[V, X] \in D$ for $V \in D^{\perp}$, we have

$$P(\nabla_V X) = P(\nabla_X V) = A_X V.$$

The operators A and C are related by

$$g(A_X V, Y) = -g(V, C(X, Y)), \quad X, Y \in D, V \in D^{\perp}. \qquad ...(2.10)$$

The curvature tensors R, $\hat{R}$ and R^* of M, the fibres and B are related by

$$R(X, Y; Z, H) = R^*(X_*, Y_*; Z_*, H_*) - g(C(X, Z), C(Y, H)) + g(C(Y, Z), C(X, H)) + 2g(C(X, Y), C(Z, H)) \qquad ...(2.11)$$

$$R(X, V; Y, W) = g(\nabla_X T)_V W, Y) + g((\nabla_V A)_X Y, W) - g(T_V X, T_W Y) + g(A_X V, A_Y W) \qquad ...(2.12)$$

$$R(U, V; W, F) = \hat{R}(U, V; W, F) - g(T_V W, T_U F) + g(T_U W, T_V F) \qquad ...(2.13)$$

for $X, Y, Z, H \in D$ and $U, V, W, F \in D^{\perp}$.

The operator C in (2.11) is introduced by Kobayashi[6] while in (2.13), the operators T and A are due to O'Neill[7] and are called the fundamental tensors of the submersion π, the operator A in (2.12) coincides with C for horizontal vector fields. The operator T for vertical vector fields will be denoted by L which we shall use in Proposition 3.3.

3. Geometry of CR-Submanifolds

In this section we study the impact of the submersion $\pi: M \to B$ on the geometry of CR-submanifold M. As a first consequence, using (2.2), (2.8) and (2.9) we get

$$C(X, JY) = Jh(X, Y)$$
$$h(X, JY) = JC(X, Y) \qquad ...(3.1)$$

from which it easily follows that

$$h(X, JY) + h(JX, Y) = 0, \quad X, Y \in D. \qquad ...(3.2)$$

Proposition 3.1—Let $\pi : M \to B$ be a submersion of CR-submanifold M of a Kaehler manifold $\bar{M}$ onto an almost Hermitian manifold B. If D is integrable and $D^{\perp}$ is parallel (i. e., $\nabla_X Y \in D$, $X, Y \in D^{\perp}$), then M is the product $M_1 \times M_2$, where M_1 is a complex submanifold and M_2 is a totally real submanifold of $\bar{M}$.

PROOF : If D is integrable, then we have from (2.4) and (3.2) $h(X, JY) = 0$, $X, Y \in D$. As a consequence of this in (3.1) we observe that $C(X, Y) = 0$, and thus $\nabla_X Y \in D$ which proves D is parallel. This completes the proof of the proposition.

Corollary 3.1—Let $\pi : M \to B$ be a submersion of a CR-submanifold M of a Kaehler manifold $\bar{M}$ with integrable D. Then

$$\bar{H}(X) = H^*(X_*), \; X \in D$$

where $\bar{H}$ and H^* are respectively the holomorphic sectional curvatures of $\bar{M}$ and B. In particular if $\bar{M}$ is of constant holomorphic sectional curvature C, then so is B. The proof follows at once from Proposition 3.1 and eqn (1.3) Kobayashi[6].

A CR-submanifold is said to be mixed foliate if D is integeable and $h(X, Y) = 0$, $X \in D, Y \in D^{\perp}$. For the submersion of fixed foliate CR-submanifolds we prove the following:

Proposition 3.2—Let $\pi : M$ be a submersion of mixed foliate CR-submanifold M of a Kaehler manifold $\bar{M}$ onto an almost Hermitian manifold B. Then M is the product $M_1 \times M_2$ where M_1 is complex submanifold and M_2 is totally real submanifold of $\bar{M}$.

PROOF : From Proposition 3.1 it follows that

$$h(X, Y) = 0, \; X, Y \in D.$$

Also M being mixed foliate we have

$$h(X, Y) = 0, \; X \in D, \; Y \in D^{\perp}.$$

Now for $X, Y \in D^{\perp}$, we have

$$(\bar{\nabla}_X J)(Y) = 0$$

i. e.

$$-\tilde{A}_{JY} X + \nabla^{\perp}_X JY = J \nabla_X Y + Jh(X, Y).$$

As $X \in D^{\perp}$, $\tilde{A}_{JY} X \in D^{\perp}$ (Bejancu[1]) and thus equating vertical components in above equation we get $-\tilde{A}_{JY} X = Jh(X, Y)$ which then gives $\overset{\perp}{\nabla}_X JY = J \nabla_X Y$, proving that

$\nabla_X Y \in D^{\perp}$ i. e. $D^{\perp}$ is parallel and thus the proof follows from Proposition 3.1.

Proposition 3.3—Let $\pi : M \to B$ be a submersion of a CR-submanifold M of a Kaehler manifold $\bar{M}$ onto an almost Hermitian manifold B. Then the fibres are totally geodesic submanifolds of M if and only if $h(X, V) = 0$, $X \in D$, $V \in D^{\perp}$.

PROOF : For $U, V \in D^{\perp}$ we define L by

$$\nabla_U V = \hat{\nabla}_U V + L(U, V)$$

where $\hat{\nabla}_U V = Q(\nabla_U V)$ and $L(U, V) = P(\nabla_U V)$. Since $D^{\perp}$ always integrable we get $L(U, V) = L(V, U)$. Now from $(\bar{\nabla}_U J)(V) = 0$ it follows that

$$-\tilde{A}_{JV} U + \nabla^{\perp}_U JV = JL(U, V) + J\hat{\nabla}_U V + Jh(U, V).$$

Equating the horizontal and vertical components we get

$$P(\tilde{A}_{JV} U) = -JL(U, V)$$

and

$$Q(\tilde{A}_{JV} U) = -Jh(U, V).$$

From this it follows that fibres are totally geodesic iff $\tilde{A}_{JV} U \in D^{\perp}$. This proves that fibres are totally geodesic iff $h(U, X) = 0 \;\forall\; X \in D, U \in D^{\perp}$.

Proposition 3.4—Let $\pi : M \to B$ be a submersion of a CR-submanifold M of a Kaehler manifold $\bar{M}$ onto an almost Hermitian manifold B. Then the sectional curvatures of $\bar{M}$ and the fibres are related by

$$\bar{K}(U \wedge V) = \hat{K}(U \wedge V) - g([\tilde{A}_{JU}, \tilde{A}_{JU}]U, V)$$

for orthonormal vector fields $U, V \in D^{\perp}$.

PROOF : We defne $\hat{R}$ by

$$\hat{R}(U, V)W = [\hat{\nabla}_U, \hat{\nabla}_V](W) - \hat{\nabla}_{[U,V]} W.$$

Now,

$$R(U, V)W = [\nabla_U, \nabla_V](W) - \nabla_{[U,V]} W$$

$$= \nabla_U \nabla_V W - \nabla_V \nabla_U W - \nabla_{[U,V]} W$$

(*equation continued on p.* 1191)

$$= \nabla_U (L(V, W) + \hat{\nabla}_V W) - \nabla_V (L(U, W) + \hat{\nabla}_U W)$$

$$- \hat{\nabla}_{[U,V]} W - P(\nabla_{[U,V]} W).$$

Taking inner product with a vertical vector field F in above relation, we get

$$R(U, V; W, E) = \hat{R}(U, V; W, F) - g(L(V, W), L(U, F)) + g(L(U, W), L(V, F)).$$

From (2.5) and above relation we have

$$\begin{aligned}\bar{R}(U, V; W, F) = \hat{R}(U, V; W, F) &- g(L(V, W), L(U, F)) \\ &+ g(L(U, W), L(V, F)) \\ &- g(h(V, W), h(U, F)) + g(h(U, W), h(V, F)).\end{aligned}$$

The above relation gives

$$\begin{aligned}\bar{R}(U, V; U, V) = \hat{R}(U, V; U, V) &- g(L(U, V), L(U, V) \\ &+ g(L(U, U), L(V, V)) \\ &- g(h(U, V), h(U, V) + g(h(U, V), h(V, V))\end{aligned}$$

which implies that

$$\begin{aligned}\bar{K}(U \wedge V) = \hat{K}(U \wedge V) &- g(L(U, V), L(U, V)) \\ &+ g(L(U, U), L(V, V)) - g(h(U, V), h(U, V)) \\ &+ g(h(U, U), h(V, V))\end{aligned}$$

for any orthonormal vectors $U, V \in D^{\perp}$.

Now using $P(\widetilde{A}_{JU} V) = -JL(U, V)$ and $Q(\widetilde{A}_{JU} V) = -Jh(U, V)$ we obtain

$$\begin{aligned}\bar{K}(U \wedge V) = \hat{K}(U \wedge V) &- g(P\widetilde{A}_{JU} V, P\widetilde{A}_{JU} V) \\ &+ g(P\widetilde{A}_{JU} U, P\widetilde{A}_{JV} V) \\ &- g(Q\widetilde{A}_{JU} V, Q\widetilde{A}_{JU} V) + g(Q\widetilde{A}_{JU} U, Q\widetilde{A}_{JV} V) \\ = \hat{K}(U \wedge V) &- g(\widetilde{A}_{JU} V, \widetilde{A}_{JU} V) + g(\widetilde{A}_{JU} V, \widetilde{A}_{JV} V) \\ = \hat{K}(U \wedge V) &- g(\widetilde{A}_{JU} V, \widetilde{A}_{JV} U) + g(\widetilde{A}_{JV} \widetilde{A}_{JU} U, V)\end{aligned}$$

(*equation continued on p. 1192*)

$$= \hat{K}(U \wedge V) - g(\tilde{A}_{JU}\tilde{A}_{JV}U, V) + g(\tilde{A}_{JV}\tilde{A}_{JU}U, V)$$

$$= \hat{K}(U \wedge V) - g([\tilde{A}_{JU}, \tilde{A}_{JV}](U), V)$$

which proves the result.

A CR-submanifold is said to be mixed totally geodesic if $h(X, Y) = 0$ for $X \in D$ and $Y \in D^{\perp}$. For mixed totally geodesic CR-submanifold we have :

Proposition 3.5 Let $\pi : M \to B$ be a submersion of a mixed totally geodesic CR-submanifold M of a Kaehler manifold $\bar{M}$ onto an almost Hermitian manifold B, then

$$\bar{R}(X, V; Y, W) = -g(\nabla_V C)(X, Y), W) - g(A_X V, A_Y W) + g(h(X, Y), h(V, W)) \qquad \ldots(3.3)$$

for $X, Y \in D$ and $V, W \in D^{\perp}$.

PROOF : From definition of R it follows that

$$\begin{aligned} R(V, X)Y &= \nabla_{[X,V]}Y - \nabla_X \nabla_V Y + \nabla_V \nabla_X Y. \\ &= P(\nabla_{[X,V]}Y) + Q(\nabla_{[X,V]}Y) - \nabla_X (P\nabla_V Y + T_V Y) \\ &\quad + \nabla_V (P\nabla_X Y + C(X,Y)) \\ &= P\nabla_{[X,V]}Y + T_{[X,V]}Y - \nabla_X (P(\nabla_V Y)) \\ &\quad - \nabla_Y (T_V Y) + \nabla_V (P\nabla_X Y) + \nabla_V C(X, Y). \end{aligned}$$

Taking inner product with $W \in D^{\perp}$ and noting that $[X, V] \in D^{\perp}$ we get

$$\begin{aligned} g(R(V, X)Y, W) &= g(T_{[X,V]}Y, W) - g(\nabla_X P(\nabla_V Y), W) \\ &\quad - g(\nabla_X (T_V Y), W) + g(\nabla_V Q(\nabla_X Y), W) \\ &\quad + g(\nabla_V C(X, Y), W) \\ &= g(T\nabla_X V\, Y, W) - g(T\nabla_V X\, Y, W) \\ &\quad + g(P\nabla_V Y, \nabla_X W) - g(\nabla_X (T_V Y), W) \\ &\quad - g(P\nabla_X Y, \nabla_V W) + g(\nabla_V C)(X, Y), W) \\ &\quad + g(C(\nabla_V X, Y), W) + g(C(X, \nabla_V Y, W) \end{aligned}$$

here $\nabla_V X \in D$ for $X \in D$ and $V \in D^{\perp}$ follows from $(\bar{\nabla}_V J)(X) = 0$ and $h(X, V) = h(JX, V) = 0$.

From definition of L in Proposition 3.3 and T in O'Neill[7] it easily follows that

$$g(T_V Y, W) = -g(L(V, W)Y).$$

Taking covariant differentiation in above equation with respect to X we obtain

$$g(\nabla_X(T_V Y), W) + g(T_V Y, \nabla_X W)$$
$$= -g(\nabla_X Y, L(V, W)) - g(\nabla_X L(V, W), Y)$$
$$= -g(\nabla_X Y, L(V, W) - g((\nabla_X L)(V, W), Y)$$
$$- g(L(Q\nabla_X V), W), Y) - g(L(V, Q(\nabla_X W), Y)$$

where we define

$$(\nabla_X L)(V, W) = \nabla_X L(V, W) - L(Q\nabla_X V, W)$$
$$- L(V, Q\nabla_X W).$$
$$= -g(\nabla_X Y, L(V, W)) - g((\nabla_X L)(V, W), Y)$$
$$+ g(T_{\nabla_X V} Y, W) + g(T_V Y, Q(\nabla_X W))$$

as T is vertical.

From which it follows that

$$g(\nabla_X(T_V Y), W) = -g((\nabla_X L)(V, W), Y) - g(\nabla_X Y, L(V, W))$$
$$+ g(T_{\nabla V} Y, W). \qquad \ldots(3.5)$$

From (3.4) and (3.5) we obtain

$$R(V, X; Y, W) = g(T_{\nabla_X V} Y, W) - g(T_{\nabla_V X} Y, W)$$
$$+ g(P(\nabla_V Y, \nabla_X W) + g(\nabla_X Y, L(V, W))$$
$$- g(T_{\nabla_X V} Y, W) + g((\nabla_X Y, L)(V, W), Y)$$
$$- g(P\nabla_X Y, \nabla_V W)$$
$$+ g((\nabla_V C)(X, Y), W) + g(C(\nabla_V X, Y) W)$$
$$+ g(C(X, \nabla_V Y), W)$$
$$= g((\nabla_X L)(V, W), Y) + g((\nabla_V C)(X, Y), W)$$
$$+ g(P\nabla_V Y, \nabla_X W) - g(T\nabla_V X\, Y, W)$$
$$+ g(C(\nabla_V X, Y), W) + g(C(X, \nabla_V Y), W).$$

Using (2.10) we get

$$R(V, X; Y, W) = g((\nabla_X L)(V, W), Y) + g((\nabla_V C)(X, Y), W)$$
$$+ g(A_Y V, A_X W) + g(Y, L(Q\nabla_V X, W))$$

(equation continued on p. 1194)

$$+ g(A_Y W, \nabla_V X) - g(A_X W, \nabla_V Y)$$
$$= g((\nabla_X L)(V, W), Y) + g((\nabla_V C)(X, Y), W)$$
$$+ g(A_Y V, A_X W) + g(T_W Y, Q\nabla_V X)$$
$$+ g(A_Y W, P\nabla_V X) - g(A_X W, P\nabla_V Y)$$
$$= g((\nabla_X L)(V, W), Y) + g((\nabla_V C)(X, Y), W)$$
$$+ g(A_Y V, A_X W) - g(T_V X, T_W Y)$$
$$+ g(A_X V, A_Y W) - g(A_X W, A_Y V)$$

where we have used definitions of A and T and $P\nabla_V X = P\nabla_X V$ as $[V, X] \in D^{\perp}$.

Thus

$$R(V, X; Y, W) = g((\nabla_X L)(V, W), Y) + g((\nabla_V C)(X, Y), W)$$
$$+ g(A_X V, A_Y W) - g(T_V X, T_W Y).$$

Now using equation of Gauss (2.5) in above equation we get

$$\bar{R}(X, V; Y, W) = -g((\nabla_X L)(V, W), Y) - g((\nabla_V C)(X, Y), W)$$
$$- g(A_X V, A_Y W) + g(T_V X, T_W Y)$$
$$- g(h(X, W), h(V, Y)) + g(h(X, Y), h(V, W)).$$

Now if M is mixed totally geodesic from Proposition 3.3 it follows that $L(V, W) = 0$ and

$$g(T_V X, T_W Y) = -g(X, L(V, Q\nabla_W Y)) = 0.$$

Hence above equation reduces to (3.3).

Proposition 3.6—Let $\pi : M \to B$ be a submersion of a mixed totally geodesic CR-submanifold M of a Kaehler manifold $\bar{M}$ onto an almost Hermitian manifold B. Then for the unit vectors $X \in D$ and $V \in D^{\perp}$ we have

$$\bar{K}(X \wedge V) = -\|\tilde{A}_{JV} X\|^2 + g(h(X, X), h(V, V)). \quad \ldots(3.6)$$

PROOF : From Proposition 3.5 for $X = Y$, $W = V$, and noting $C(X, X) = 0$ we get

$$\bar{K}(X \wedge V) = g(h(X, X), h(V, V)) - \|A_X V\|^2. \quad \ldots(3.7)$$

Using $(\bar{\nabla}_X J)(V) = 0$ we get

$$-\tilde{A}_{JV} X + \overset{\perp}{\nabla}_X JV = JA_X V + JQ\nabla_X V + Jh(X, V).$$

Since M is mixed-totally geodesic $h(X, V) = 0$ and this implies $\tilde{A}_{JV} X \in D$ for every $X \in D$. Thus equating horizontal component in above equation we get $J A_X V = - \tilde{A}_{JV} X$ or

$$A_X V = J\tilde{A}_{JV} X. \qquad \ldots(3.8)$$

Using (3.8) in (3.7) we get the result.

Proposition 3.7—If $\pi : M \rightarrow B$ is a submersion of a mixed foliate CR-submanifold M of a Kaehler manifold $\bar{M}$ onto an almost Hermitian manifold B, then the curvature tensor $\bar{R}$ of $\bar{M}$ satisfies

$$\bar{R}(X, V; Y, W) = 0, X, Y \in D, V, W \in D^{\perp}.$$

PROOF : If M is foliate, then from Proposition 3.1, it follows that $h(X, Y) = C(X, Y) = 0$. Also

$$g(A_X V, A_Y W) = - g(C(X, A_Y W), V) = 0.$$

Using this in Proposition 3.5 we get the result.

Following Bejancu[2], we say that the normal connection $\nabla^{\perp}$ is D-flat if $R^{\perp}(X, Y; N, N') = 0$, $X, Y \in D$. Now we are in position to prove our main theorem.

Theorem 3.1—If $\pi : M \rightarrow B$ is a submersion of a mixed foliate CR-submanifold M of a Kaehler manifold $\bar{M}$ onto an almost Hermitian manifold B, then the normal connection of M in $\bar{M}$ is D-flat.

PROOF : Using Proposition (3.7) in the Bianchi's identity

$$\bar{R}(X, V; Y, W) + \bar{R}(V, Y; X, W) + \bar{R}(Y, X; V, W) = 0$$

we get

$$\bar{R}(X, Y; V, W) = 0, X, Y \in D, V, W \in D^{\perp}.$$

Now using $\bar{R}(X, Y, V, W) = \bar{R}(X, Y, JV, JW)$ we get

$$\bar{R}(X, Y; JV, JW) = 0.$$

Using the equation of Ricci (2.7) we get

$$R^{\perp}(X, Y; N, N') = g([\tilde{A}_N, \tilde{A}_{N'}](X), Y)$$

where $JV = N$ and $JW = N'$ are normals.

Thus

$$R^{\perp}(X, Y; N, N') = g(\tilde{A}_N \tilde{A}_{N'} X, Y) - g(\tilde{A}_{N'} \tilde{A}_N X, Y)$$

(*equation continued on p.* 1196)

$$= g(\widetilde{A}_{JW}\ X, \widetilde{A}_{JV}\ Y) - g(\widetilde{A}_{JV}\ X, \widetilde{A}_{JW}\ Y)$$

$$= g(A_X W, A_Y V) - g(A_X V, A_Y W)$$

where we have used (3.8). As in proof of (3.7), we get $g(A_X W, A_Y V) = g(A_X V, A_Y W) = 0$. Hence, we get the result.

4. Submersions of Totally Umbilical CR-submanifolds

In last section we have discussed those submersions of CR-submanifolds of a Kaehler manifold in which mostly the CR-submanifolds were turning to be totally geodesic and as such $\overline{M}$ and B were becoming isocurved (cf. Propositions 3.1, 3.2). Next natural question is which non-totally geodesic CR-submanifolds maintain this property of $\overline{M}$ and B being specially the spaces of constant holomorphic curvature. Very natural non-totally geodesic CR-submanifolds are totally umbilical CR-submanifolds of Kaehler manifolds, moreover, they are natural prototypes for the submersion $\pi : M \to B$, because the condition (3.2) is naturally satisfied for $h(X, JY) = g(X, JY) H$ and $h(JX, Y) = g(JX, Y)H$, where H is mean curvature vector which is non-zero for non-totally geodesic submanifolds. In case of totally umbilical CR-submanifolds the equations (2.2), (2.3) and (2.6) take the following forms

$$\overline{\nabla}_X Y = \nabla_X Y + g(X, Y) H \qquad \text{...(4.1)}$$

$$\nabla_X N = - g(N, H) X + \nabla^{\perp}_X N. \qquad \text{...(4.2)}$$

$$[\bar{R}(X, Y) Z]^{\perp} = g(Y, Z) \nabla^{\perp}_X H - g(X, Z) \nabla^{\perp}_Y H. \qquad \text{...(4.3)}$$

In case $\overline{M}$ is complex space form of constant holomorphic sectional curvature c, the curvature tensor $\bar{R}$ is given by

$$\begin{aligned}\bar{R}(X, Y; Z, W) = c/4\ [&g(Y, Z) g(X, W) - g(X, Z) g(Y, W) \\ &+ g(JY, Z) g(JX, W) - g(JX, Z) g(JY, W) \\ &+ 2g(X, JY) g(JZ, W)].\end{aligned} \qquad \text{...(4.4)}$$

Our main theorem in this section is

Theorem 4.1—Let $\pi : M \to B$ be the submersion of a totally umbilical CR-submanifold M ($\dim M \geqslant 5$) of a complex space form $\overline{M}(c)$ onto an almost Hermitian manifold B. Then B is also a complex space form.

Proof : Since in case of submersion $\pi : M \to B$ $JD^{\perp} = \nu$, from Theorem (cf. Blair and Chen)[3] it follows that either $H = 0$ or $\dim D^{\perp} = 1$. In case $H = 0$ from eqn. (1.3) of Kobayashi[6] (Theorem 1.3) it follows that B is also a complex space form.

Suppose dim $D^\perp = 1$. From eqns. (2.5), (2.11), (4.4) and $h(X, Y) = g(X,Y)H$, we easily get the following expression for the curvature tensor R^* of B.

$$R^*(X_*, Y_*; Z_*, W_*) = (c/4 + \|H\|^2)\{g(Y, Z)\, g(X, W) - g(X,Z)\, g(Y,W) + g(JY, Z)\, g(JX, W) - g(JX, Z)\, g(JY, W) + 2g(X, JY)\, g(JZ, W)\}.$$

Thus to complete the proof we have to show that $\|H\|^2$ is a constant. Since dim $M \geqslant 5$ we can choose vectors $X, Y \in D$ such that $g(X, Y) = g(X, JY) = 0$. Now from equation (2.6) of Codazzi we have

$$\bar{R}(X, Y; Z, N) = g(Y, Z)\, g(\nabla^\perp_X H, N) - g(X, Z)\, g(\nabla^\perp_Y H, N).$$

From (4.4) it follows that $\bar{R}(JY, X; JY, N) = 0$. Thus (4.3) gives

$$g(\nabla^\perp_X H, N) = 0,\ N \in \nu. \qquad \ldots (4.5)$$

This proves that

$$\nabla^\perp_X H = 0 \ \forall\ X \in D.$$

Next let $X \in D^\perp$. Then using the following curvature properties of $\bar{M}$

$$\bar{R}(JX, JY; JZ, W) = \bar{R}(X, Y, Z, W)$$

$$\bar{R}(JX, JY; Z, W) = \bar{R}(X, Y; Z, W)$$

and (4.4) we get

$$\bar{R}(X, Y; Y, X) = \bar{R}(X, Y; JY, N') = 0,\ N' = JX.$$

Using linearity of $\bar{R}$ in $\bar{R}(X, Y; Y, X) = 0$ we get

$$\bar{R}(X, Y; JY, X) = 0$$

or

$$\bar{R}(X, Y; Y, N') = 0.$$

Using this in (4.3) we get

$$g(\nabla^\perp_X H, N') = 0.$$

As dim $D^\perp$ = dim $\nu = 1$, we get $\nabla^\perp_X H = 0$ for $X \in D^\perp$. Hence for any vector field X on M we have

$X. \|H\|^2 = X. g(H, H) = 2g(\nabla_X^{\perp} H, H) = 0$, proving that $\|H\|^2$ = constant and hence the theorem.

Theorem 4.2—Let $\pi : M \to B$ be a submersion of a totally umbilical CR-submanifold M of a Kaehler manifold $\bar{M}$ with parallel D. Then M is the product $M_1 \times M_2$ where M_1 is complex submanifold and M_2 is totally real submanifold of $\bar{M}$.

PROOF : Let H be the mean curvature vector of the CR-submanifold M in $\bar{M}$. Since H is normal, JH is vertical.

Using Gauss and Weingarten formulae in $(\bar{\nabla}_X J)(JH) = 0$, obtain

$$\tilde{A}_H X - \nabla_X^{\perp} H = J \nabla_X JH + Jh(X, JH).$$

Now using the definition of totally umbilicalness in above relation we get

$$g(H, H) X - \nabla_X^{\perp} H = J \nabla_X JH + h(X, JH) JH. \quad \ldots(4.6)$$

Taking inner product with $X \pm 0 \in D$ in (4.6) we get

$$\begin{aligned} \|H\|^2 \|X\|^2 &= -g(\nabla_X JH, JX) \\ &= g(JH, \nabla_X JX) \end{aligned} \quad \ldots(4.7)$$

As D is parallel, $\nabla_X JX \in D$, which implies that $g(\nabla_X JX, JH) = 0$, the above relation (4.7) gives $\|H\|^2 = 0$, i. e M is totally geodesic and hence the result.

Remark : Wherever necessary, the horizontal vector fields are supposed to be basic.

5. Ricci Tensors and Scalar Curvature

In this section we obtain relations between the Ricci tensors and scalar curvatures of $\bar{M}$ and the base manifold. We have

Theorem 5.1—Let $\pi : M \to B$ be a submersion of a mixed foliate CR-submanifold M of a Kaehler manifold $\bar{M}$ onto an almost Hermitian manifold B. Then the Ricci tensors $\bar{S}$ and S^* of $\bar{M}$ respectively B satisfy the relation

$$\bar{S}(X, Y) = S^*(X_*, Y_*) \quad \ldots(5.1)$$

for each basic vector fields $X, Y \in D$.

PROOF : From (2.5) and (2.11)

$$\begin{aligned} \bar{R}(Z, X, Y, W) = {} & R^*(Z_*, X_*, Y_*, W_*) + g(h(Y, Z), h(X, W)) \\ & - g(h(W, Z), h(X, Y)) - g(C(X, Y), C(Z, W)) \\ & + g(C(Z, Y), C(X, W)) + 2g(C(Z, X), C(Y, W)). \end{aligned} \quad \ldots(5.2)$$

Let $\{E_1, \ldots, E_{2p}, E_{p+1} = JE_1, \ldots, E_{2p} = JE_p, F_1, \ldots, Fq, JF_1, \ldots JF_q\}$ be a lacol field of orthonormal frames of $\bar{M}$ such that $\{E_1, \ldots, E_p, E_{p+1} = JE_1, \ldots, E_{2p} = JE_p\}$ *and* $\{F_1, \ldots, F_q\}$ are local fields of orthnormal frames of the horizontal distribution D and the vertical distribution $D^{\perp}$ respectively. Then using the definition of of the Ricci tensor in (5.2) we obtain

$$\bar{S}(X, Y) = S^*(X_*, Y_*) + \sum_{i=1}^{2p} g(h(E_i, Y), h(E_i, X)) \qquad \ldots(5.3)$$

$$- g(h(X, Y), \sum_{i=1}^{2p} h(E_i, E_i)) - g(C(X, Y), \sum_{1=1}^{2p} C(E_i, E_i))$$

$$+ \sum_{i=1}^{2p} \{g(C(E_i, Y), C(X, E_i))\} + 2 \sum_{i=1}^{2p} \{g\, C(E_i, X),$$

$$C(Y, E_i))\}$$

$$+ \sum_{k=1}^{q} \{\bar{R}(F_k, X; Y, F_k) + \bar{R}(JF_k, X; Y, JF_k)\}.$$

Since M is foliate, the horizontal distribution D is involutive, then M is D-minimal[1]. On the other hand C is skew symmetric. Hence the third and fourth term on the right-hand side of (5.3) vanishes and we get

$$\bar{S}(X, Y) = S^*(X_*, Y_*) + \sum_{i=1}^{2p} g(h(E_i, y), h(E_i, X))$$

$$- 3 \sum_{i=1}^{2p} g(C(E_i, X), C(E_i, Y)) \qquad \ldots(5.4)$$

$$+ \sum_{k=1}^{q} \bar{R}(F_k, X; Y, Y_k) + \bar{R}(JF_k, X; Y, JF_k).$$

Now, from $g(h(X, Y), N) = g(A_N X, Y)$ and Lemma 2.1[2] we get

$$\sum_{k=1}^{2p} \{g(h(E_i, X), h\, E_i, Y))\} \qquad \ldots(5.5)$$

$$= \sum_{k=1}^{q} g(\widetilde{A}_{JF_k} X, \widetilde{A}_{JF_k} Y).$$

Also we have[6] $C(X, JY) = Jh(X, Y)$, which implies

$$\sum_{i=1}^{2p} g(C(E_i, X), C(E_i, Y)) = \sum_{i=1}^{2p} g(-Jh(E_i, JX), -Jh(E_i, JY))$$

(*equation continued on p.* 1200)

$$= \sum_{i=1}^{2p} g\,(h\,(E_i, JX, h\,(E_i, JY))$$

$$= \sum_{k=1}^{q} g\,(\widetilde{A}_{JF_k}\; JX,\; \widetilde{A}_{JF_k}\; JY) \qquad \ldots(5.6)$$

$$= \sum_{k=1}^{q} g\,(-J\widetilde{A}_{JF_k}\; X, -J\widetilde{A}_{JF_k}\; Y)$$

$$= \sum_{k=1}^{q} g\,(\widetilde{A}_{JF_k}\; X,\; \widetilde{A}_{JF_k}\; Y).$$

Using eqn. (3.8) we obtain

$$\sum_{k=1}^{q} g\,(\widetilde{A}_{JF_k}\; X,\; \widetilde{A}_{JF_k}\; Y) = \sum_{k=1}^{q} g\,(A_X\, F_k, A_Y\, F_K). \qquad \ldots(5.7)$$

If M is foliate, then from Proposition 3.1, it follow that $h\,(X, Y) = C\,(X, Y) = 0$. Also

$$g\,(A_X\, F_K,\; A_Y\; F_K) = -\, g\,(C\,(X, A_Y\, F_k),\; F_k) = 0. \qquad \ldots(5.8)$$

Using (5.5), (5.6), (5.7), and (5.8) in (5.4) we get

$$\bar{S}\,(X, Y) = S^*\,(X_*, Y_*) + \sum_{k=1}^{q} R\,(F_K, X; Y, F_K) + R\,(JF_k, X, Y, JF_k). \qquad \ldots(5.9)$$

From Proposition 3.7.

$$\bar{R}\;(F_K, X, Y,\; F_K) = 0 \;\forall\; X, Y \in D. \qquad \ldots(5.10)$$

Also, Bianchi identity gives

$$R\,(JF_K, X, Y; JF_K) + \bar{R}\,(X, Y, JF_K, JF_K) + \bar{R}\,(Y, JF_K, X, JF_K) = 0.$$

Using Theorem 3.1 and Proposition 3.7 we get

$$\bar{R}\;(JF_K, X; Y, JF_K) = 0 \qquad \ldots(5.11)$$

From (5.9), (5.10) and (5.11) we get the result.

Definition 5.1—The Kaehler manifold $\overline{M}$ is said to be an Einstein space if there exists a constant σ such that the Ricci tensor $\bar{S}$ of $\overline{M}$ satisfies

$$\bar{S}\,(X, Y) = \sigma g\,(X, Y) \qquad \ldots(5.12)$$

for all tangent vectors X, Y on $\overline{M}$.

As a direct consequence of (5.12) and above theorem, we have

Theorem 5.2—Let M be a mixed foliate CR-submanifold of a Kaehler manifold

$\overline{M}$ and let $\pi : M \to B$ be a submersion of M onto an almost Hermitian manifold B. Then B is an Einstein space if and only if $\overline{M}$ is an Einstein space.

Lastly in this section we estimate the Ricci tensor and scalar curvature of the base manifold B of the submersion $\pi : M \to B$ of M onto an almost Hermitian manifold B, when M is a CR-submanifold of a complex space form $\overline{M}(c)$ of constant holomorphic sectional curvature c.

Let $\{E_m, \ldots, E_m\}$ be a local field of orthonormal frames on M (where m is the dimension of the CR-submanifold M) such that $\{E_1, \ldots, E_p, E_{p+1} = JE_1, \ldots, E_{p_2} = JE_p\}$ is a local field of orthonormal frames on D and $\{F_1, \ldots, F_q\}$ is a local field of orthonormal frames on $D^{\perp}$. Then $\{JF_1, \ldots, JF_q\}$ becomes the field of orthonormal frames of the normal bundle ν.

Then from (5.2) and (4.5) we obtain

$$\begin{aligned} R^*(Z_*, X_*; Y_*, W_*) = {} & c/4\, g(X, Y)\, g(Z, W) - g(Z, Y)\, g(X, W) \\ & + g(JX, Y)\, g(JZ, W) - g(JZ, Y)\, g(JX, W) \\ & + 2g(Z, JX)\, g(JY, W) \\ & + g(h(Z, W), h(X, Y)) - g(h(Z, Y), h(X, W)) \\ & + g(C(X, Y), C(Z, W)) - g(C(Z, Y), \\ & \times C(X, W)) - 2g(C(Z, X), C(Y, W)) \end{aligned} \quad \ldots(5.13)$$

for any basic vector fields X, Y, Z, W on M.

Then from (5.13) we get

$$\begin{aligned} S^*(X_*, Y_*) = {} & c/4\, [2p.\, g(X, Y) - \sum_{i=1}^{2p} \{g\, E_i, Y)\, g(X, E_i)\} \\ & + \sum_{i=1}^{2p} \{g(JX, Y)\, g(JE_i, E_i)\} - \sum_{i=1}^{2p} \{g(JE_i, Y)\, g(JX, E_i)\} \\ & + 2 \sum_{i=1}^{2p} \{g(E_i, JX)\, g(JY, E_i)\}] + g(h(X, Y), \sum_{i=1}^{2p} h(E_i E_i)) \\ & - \sum_{i=1}^{2p} g(h(E_i, Y), h(X, E_i)) + g(C(X, Y), C(E_i, E_i))\} \\ & - \sum_{i=1}^{2p} \{g(C(E_i, Y)\, C(X, E_i)\} - 2 \sum_{i=1}^{2p} \{g(C)\, E_i, X), \\ & C(Y, E_i))\}. \end{aligned} \quad \ldots(5.14)$$

Using the skew-symmetry of C, we get

$$S^* (X_*, Y_*) = \frac{pc}{2} g(X, Y) - \frac{c}{4} \sum_{i=1}^{2p} \{g(E_i, Y) g(X, E_i)$$

$$- 3g(JE_i, Y) g(JX, E_i)\} + g(h(X, Y), \sum_{i=1}^{2p} h(E_i, E_i))$$

$$- \sum_{i=1}^{2p} \{g(h(E_i, Y), h(X, E_i))\}$$

$$+ 3 \sum_{i=1}^{2p} g(C E_i, Y), C(E_i, X))\}. \quad \text{..(5.15)}$$

Now we have the following equation (4.1), (4.2), (4.3) of Bejancu[1]

$$\sum_{i=1}^{m} \{g(JPE_i, Y) g(JPX, E_i)\} = - g(PX, PY)$$

$$\sum_{i=1}^{m} \{g(JPE_i. E_i)\} = 0$$

$$\sum_{i=1}^{m} \{g(E_i, JPX) g(E_i, JPY)\} = g(PX, PY),$$

for any vector field X, Y on M, so in our case the above equations take the following forms.

$$\sum_{i=1}^{2p} \{g(JE_i, Y) g(JX, E_i)\} = - g(X, Y) \quad \text{...(5.16)}$$

$$\sum_{i=1}^{2p} \{g(JE_i, E_i)\} = 0 \quad \text{...(5.17)}$$

$$\sum_{i=1}^{2p} \{g(E_i, JX) g(E_i, JY)\} = g(X, Y). \quad \text{...(5.18)}$$

If we use (5.16) – (5.18) in (5.15) we obtain following expression for the Ricci tensor S^* of B

$$S^* (X_*, Y_*) = \frac{pc}{2} g(X, Y) - \frac{c}{4} g(X, Y) + \frac{3c}{4} g(X, Y)$$

$$+ \sum_{i=1}^{2p} \{g(h(X, Y), h(E_i, E_i) - g(h(E_i, Y), h(E_i, X))\}$$

$$+ 3 \sum_{i=1}^{2p} \{g(C(E_i, Y), C(E_i, X))\}$$

(equation contiuned on p. 1203)

$$= \frac{(p+1)c}{2} g(X, Y) + \sum_{i=1}^{2p} \{g(h(X, Y), h(E_i, E_i))$$

$$- g(h(E_i, X), h(E_i, X))\} + 3 \sum_{i=1}^{2p} \{g(C(E_i, Y), C(E_i, X))\}$$

$$S^*(X_*, Y_*) = \frac{(p+1)c}{2} g(X, Y) + \sum_{i=1}^{2p} \{g(h X, Y), h(E_i, E_i)$$

$$- g(h(E_i, Y)\}, h(E_i, X)\} + 3 \sum_{i=1}^{2p} \{g(h(E_i, J_U Y), h(E_i, JX))\}$$

$$[\because C(C(E_i, Y) = - Jh(E_i, JY)]. \qquad ...(5.19)$$

If we compute the scalar curvature ρ^* of B, we get

$$\rho^* = p(p+1)c + \sum_{i,j=1}^{2p} \{g(h(E_i, E_i), h(E_j, E_j)$$

$$- g(h(E_i, E_j), h(E_i, E_j)\} \qquad ...(5.20)$$

$$+ 3 \sum_{i=1}^{2p} \{g(h(E_i, JE_j), h(E_i, JE_j\}.$$

Theorem 5.1—Let $\pi : M \to B$ b be a submersion of a mixed foliate CR-submanifold M of complex space form $\overline{M}(c)$ of constant holomorphic sectional curvature c, onto an almost Hermitian manifold B. Then the Ricci tensor S^* of B satisfies:

$$S^*(X_*, Y_*) = \frac{(p+1)c}{2} g(X, Y)$$

for any horizontal vector field $X, Y \in D$. Where 2p is the dimension of D.

As a direct consequence of above theorem we have

Corollary 5.1—Under the hypothesis of above theorem, B is an Einstein space.

PROOF OF THEOREM 5. 1 : Since $\{JF_1, \ldots, JF_q\}$ is a local field of orthonormal frames of the normal bundle ν. We have

$$h(X, Y) = \sum_{k=1}^{q} g(\widetilde{A}_{JF_k}, X. Y) JF_k.$$

Using above relation we obtain

$$\sum_{i=1}^{2p} \{g(h(X, Y), h(E_i, E_i))\} = \sum_{k=1}^{q} (\text{tr. } \widetilde{A}_K) g(\widetilde{A}_{JF_k} X, Y) \qquad ...(5.21)$$

$$[\widetilde{A}_k = \widetilde{A}_{JF_k}]$$

$$\sum_{i=1}^{2p} \{g(h(E_i, X), h(E_i, Y)\} = \sum_{k=1}^{q} g(\widetilde{A}_{JF_k} X, \widetilde{A}_{JF_k} Y) \qquad \ldots(5.22)$$

$$\sum_{i=1}^{2p} \{g(h(E_i, JX), h(E_i, JY)\} = \sum_{k=1}^{q} g(\widetilde{A}_{JF_k} JX, \widetilde{A}_{JF_k} JY)$$

$$= \sum_{k=1}^{q} g(\widetilde{A}_{JF_k} X, \widetilde{A}_{JF_k} Y) \qquad \ldots(5.23)$$

$[\therefore M$ is mixed foliate$]$

Using (5.19), (5.21), (5.22) and (5.23) we get

$$S^*(X_*, Y_*) = \frac{(p+1)c}{2} g(X, Y) + \sum_{k=1}^{q} 2g(\widetilde{A}_{JF_k} X, \widetilde{A}_{JF_k} Y)$$

$$= \frac{(p+1)}{2} cg(X, Y) + 2 \sum_{k=1}^{q} g(A_X F_k, A_Y F_k).$$

In the proof of Proposition 3.7 we have $g(A_X V, A_Y W) = 0$, hence the above relation transforms into

$$S^*(X_*, Y_*) = \frac{(p+1)c}{2} g(X, Y)$$

which gives the result.

Theorem 5.2—Let M be a foliate CR-submanifold of a complex space form $\overline{M}(c)$ of constant holomorphic sectional curvature c. Let $\pi : M \to B$ be a submersion of M onto an almost Hermitian manifold B. Then M is D-totally geodesic if and only if the scalar curvature of B satisfies

$$\rho^* = p(p+1)c$$

PROOF : Since M is foliate, D is integrable, $h(E_i, JE_j)) = Jh(E_i, E_j))$ and also M is D minimal[1]. Therefore from (5.20) we get

$$= p(p+1)c + 2 \sum_{i=1}^{2p} \|h(E_i, E_j)\|^2$$

which proves the theorem.

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STABLE AND PSEUDO STABLE NEAR RINGS

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In this paper, we introduce the concepts of (i) Stable near rings, (ii) Pseudo Stable near rings and (iii) Mate functions in a near ring. We make use of (iii) to discuss the properties of (i) and (ii). We obtain necessary and sufficient conditions (a) for Stable and Pseudo-Stable-near rings to be near-fields and (b) for Pseudo Stable near rings to be Stable.

1. INTRODUCTION

A near ring $(N, +, .)$—more precisely a right near ring—is an algebraic system with two binary operations such that (i) $(N,+)$ is a group—not necessarily abelian (with 0 as its identity element) (ii) $(N,.)$ is a semigroup (we write xy instead of $x \,.\, y$ for all x, y in N) and (iii) $(a + b)\, c = ac + bc$ for all a, b, c in N. Throughout this paper, N stands for a near ring with atleast two elements. E denotes the set of all idempotents and L is the set of all nilpotent elements of N.

$N_d = \{n \in N/n\,(x + y) = nx + ny$ for all x, y in $N\}$ and $N_0 = \{n \in N/n0 = 0\}$. N is zero-symmetric if $N = N_0$. We write $N = N_0\,(0)$ when $N = N_0$ and $L = \{0\}$. If S is any non-empty subset of N, then (i) $C\,(S) = \{n \in N/ns = sn$ for all s in $S\}$ (We write $C\,(x)$ for $C\,(S)$ when $S = \{x\}$) and (ii) if $0 \in S$, $S^* = S - \{0\}$.

Basic concepts for a near ring and terms used but left undefined in this paper can be found in Pilz[4]. In this paper, all near field are zero-symmetric.

2. STABLE AND PSEUDO STABLE NEAR RINGS

Definition 2.1—We define N to be Stable if for all x in N, $xN = xNx = Nx$.

Examples 2.1.1—(a) A near field is obviously Stable. (b) The direct sum of a near field with itself is a Stable near ring.

Remark 2.1.2 : The Definition 2.1 demands that a stable near ring is zero-symmetric.

Remark 2.1.3: We consider the following near rings which can be easily obtained from any given group $(N, +)$:

(i) The trivial near ring $(N, +, .)$, with $a.b = 0$ for all a, b in N, certainly satisfies the Definition 2.1. But it is too trivial to be of any interest.

(ii) The constant near ring $(N, +, .)$ has the semigroup operation '.' defined as follows: $a. b = a$ for all a, b in N (or equivalently $a.0 = a$ for all a in N). Since our assumption is that N has atleast two elements, it is easy to observe that a constant near ring is an example of a near ring which is not stable.

(iii) When $N \neq Z_2$, the near ring $(N, +, .)$ with "$a . b = a$ for all a in N and for all b in N^* and $a.0 = 0$" also serves as an example of a near ring which is not Stable.

The examples (ii) and (iii) above will serve as easy examples of a more general structure to follow.

Remark 2.1.4 : If N is Stable, it is readily Subcommutative i. e. $xN = Nx$ for all x in N. We show by an example that the converse is not true in general :

Exampie 2.1.5—Let $(N, +)$ be the familiar group of integers modulo 8. We define '.' in N as follows as per scheme (48), p.343 of Pilz[4]

.	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	2	4	2	0	2	4	6
2	0	4	0	4	0	4	0	4
3	0	6	4	6	0	6	4	2
4	0	0	0	0	0	0	0	0
5	0	2	4	2	0	2	4	6
6	0	4	0	4	0	4	0	4
7	0	6	4	6	0	6	4	2

$(N, +, .)$ is a near ring which is Subcommutative but not Stable.

Definition 2.2—We define N to be Pseudo Stable (Reverse Pseudo Stable) if $aN = bN \Rightarrow Na = Nb$ ($Na = Nb \Rightarrow aN = bN$) for a, b in N.

We write 'N has (PS)' ('N has (RPS)') whenever N is Pseudo Stable (Reverse Pseudo Stable).

Remark 2.2.1 : The concept of 'Pseudo Stability' is a generalization of the concept of 'Stability' in a near ring. We furnish below the motivation for such a generalization :

Let S be a subgroup of $(N, +)$. Then the following statements are equivalent : (i) $n + S = S + n$ for all n in N. (ii) $a + S = b + S \Rightarrow S + a = S + b$ for a, b in N.

(i) is the familiar condition for S to be normal in $(N, +)$ and obviously (i) $\Rightarrow$ (ii). To prove (ii) $\Rightarrow$ (i), we observe that for every s in S and for every n in N, $n + S = n + s + S \Rightarrow S + n = S + n + s \Rightarrow S = S + (n + s - n)$ and (i) follows since $S = S + x$ iff $x \in S$. The condition (ii) can be replaced by the equivalent condition : '(iii) $S + a = S + b \Rightarrow a + S = b + S$ for a, b in N'.

Clear as it is, these equivalent "normality conditions" are the motivating forces for the Definition 2.2.

Remark 2.2.2—'Pseudo stability' and 'reverse pseudo stability' are two different concepts and neither implies the other in a near ring, in general. Also neither of them nor their combination will imply stability in general. But it is obvious that when N is Stable, it has both (PS) and (RPS).

Examples 2.2.3—(i) Consider the near rings $(N, +, *)$ and $(N, +, .)$ defined on the Klein's four group $(N, +)$ with $N = \{0, a, b, c\}$, where $*$ and $.$ are defined as follows (as per schemes (11) and (20), p. 340 of Pilz[4] and these form part of Clay[2]).

$*$	0	a	b	c
0	0	0	0	0
a	0	a	b	a
b	0	0	0	0
c	0	a	b	a

$.$	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	a	b	c
c	a	0	c	b

$(N, +, *)$ has (RPS) but is not Pseudo stable. $(N, +, .)$ has (PS) but is not Reverse Pseudo Stable. Neither of them is Stable.

(ii) The near ring given in the Example 2.1.5 has both (PS) and (RPS) but is not Stable.

(iii) The Examples cited in (ii) and (iii) under the Remark 2.1.3 are trivial examples of a near ring with (PS). They are neither reverse pseudo stable nor stable.

Lemma 2.3—If $xy = 0$ for some x, y in N, then $(yx)^r = y0$ for every integer $r \geqslant 2$. If $N = N_0\ (0)$, then $xy = 0 \Rightarrow yx = 0$ and N has Insertion of Factors Property (IFP).

PROOF : $xy = 0 \Rightarrow (yx)^2 = yxyx = y0 \Rightarrow (yx)^r = yxyx \ldots r$ times (for all integes $r \geqslant 2$) $= y0$. Also when $N = N_0\ (0)$, $xy = 0 \Rightarrow (yx)^2 = 0 \Rightarrow yx = 0$. Further, for every n in N, $(xny)^2 = xnyxny = xn\,0 = 0 \Rightarrow xny = 0$. Hence N has IFP.

Lemma 2.4—If N is Stable, $E \subseteq C(N)$.

PROOF : When N is Stable, we have in particular $eN = eNe = Ne$ for all e in E. Clearly then, for every n in N, there exist u and v in N such that $en = eue$ and $ne = eve$. It follows that $en\ (= ene) = ne$. Hence the result.

Remark 2.4.1 : The converse of Lemma 2.4 is not true. The near ring $(N, +, .)$ of Example 2.1.5 comes in handy to justify this. As indicated already, N is not Stable, but "$E = \{0\}$ and $N = N_0$" guarantee "$E \subseteq C(N)$".

Lemma 2.5—Let $a^2 = ba$ and $b^2 = ab$ for a, b in N. Let $u_1 = a - b$, $u_2 = au_1$ and $u_3 = bu_1$. If there exist x_i 's in N such that $u_i = x_i\, u_i^2$ $(i = 1, 2, 3)$, then $a = b$.

PROOF : We have $u_1 a = 0 = u_1 b$. By Lemma 2.3, we have $u_2^2 = (au_1)^2 = a0$ and $u_3^2 = (bu_1)^2 = b0$. Also $u_2 = x_2\, u_2^2 = x_2 a0$ and hence $u_2 = u_2^2 = a0$. Similarly $u_3 = u_3^2 = b0$. We observe that $u_1^2 = u_2 - u_3 = u_1 0$. Again, as $u_1 = x_1\, u_1^2$, we have $u_1 = u_1^2 = u_1 0$. Therefore, $u_1 a = u_1 0\, a = u_1 0 = u_1$. Since $u_1 a = 0$ we get the desired result.

For our further discussion we need the concept of a mate function in a near ring.

3. Mate Functions

We introduce the concept of 'mate functions' with a view to enable us to deal with the regularity structure in a near ring with considerable ease.

Definition 3.1—Let there exist a map $m : N \to N$ such that $a = a\, m(a)\, a$ for all a in N. We call m, a mate function for N. $m(a)$ is called a mate of a.

Remark 3.1.1. : The above definition guarantees the following : (a) If N has a mate function, N must naturally be regular. (b) If N has one mate function, it has many more, since every element of N can serve as a mate of O.

Examples 3.1.2—The near ring $(N, +, .)$ given as an example (of a Pseudo Stable near ring) in 2.2.3 admits mate functions. The maps m and g defined on this near ring by $m(0) = a$, $m(a) = 0$, $m(b) = b$, $m(c) = c$ and $g(0) = c$, $g(a) = b$, $g(b) = b$, $g(c) = c$ respectively are mate functions for N. The identity map is also a mate function for N. It can easily be verified that this near ring has exactly sixteen mate functions.

Lemma 3.2—If N has a mate function m, then (i) $m(a)\, a$ and $am(a)$ are idempotents. (ii) $Na = Nm(a)\, a$ and (iii) $aN = a\, m(a)\, N$ for every a in N.

PROOF : (i) is a consequence of 'Definition 3.1'

To prove (ii), we need only to observe that

$$Na = Na\, m\,(a)\, a \subseteq N\, m\,(a)\, a \subseteq Na.$$

and (iii) follows in a similar fashion.

Remark 3.2.1 : This Lemma will be made use of throughout this paper. Mainly due to this and partly because of the fact that we do not demand $(N, .)$ to have identity-one sided or two sided—we have chosen to prove this Lemma separately though it forms part of the following theorem :

Theorem 3.3—Let N have a left (right) identity. Then a map m from N into N is a mate function for N iff $m\,(a)\,a\,(a\,m\,(a))$ is an idempotent and $Na = Nm\,(a)\,a\ (aN = a\,m\,(a)\,N)$ for all a in N.

PROOF : The necessity of the condition follows from Lemma 3.2. For the sufficiency part, we observe that as N has a left identity, $x \in N \Rightarrow x \in Nx = Nm\,(x)\,x$. This demands that for every x in N, there exists some n in N such that $x = nm\,(x)\,x$. Since $m\,(x)\,x \in E$, we have $xm\,(x)\,x = n\,m\,(x)\,x = x$ and the desired result follows.

As an immediate consequence of the above theorem we have :

Corollary 3.4—Let N have the identity element. Then a map $m : N \to N$ is a mate function for N iff the conditions of Theorem 3.3—either for the case when $(N, .)$ has a left identity or for the case when it has a right identity—are satisfied.

The Definition 3.1 does not guarantee that $m\,(x) = m\,(x)\,x\,m\,(x)$ i. e. x need not be a mate of $m\,(x)$—(where m is a mate function for N)—for any x in N. But we shall show that if N admits a mate function m, then m gives rise to a mate function g, possibly different from m, such that x and $g\,(x)$ are mates of each other. Before that, we have the following :

Definition 3.5—A mate function m of N is defined to be a mutual mate function, if x is also a mate of $m\,(x)$ for every x in N. We refer to each of x and $m\,(x)$ as a mutual mate of the other. If a mutual mate function m happens to be an involution, we call m an involutory mate function for N.

Remark 3.5.1: (a) For a mate function m of N to be a mutual mate function for N, we just demand that x and $m\,(x)$ are mutual mates for every x in N. x need not be the mate of $m\,(x)$ under the same m. (b) It is obvious that every 'involutory mate function' is a 'mutual mate function' but not conversely. In the Examples 3.1.2, $(N, +, .)$ has m and the identity map as involutory mate functions. If $h : N \to N$ is such that h and m agree in N^* and $h\,(0) = 0$, then h is a mutual mate function (but not an involutory one) for N. The mate function g is not a mutual mate function for N.

Lemma 3.6—If N has a mate function m, it certainly has a mutual mate function.

PROOF : Define $f : N \to N$ such that $f\,(x) = m\,(x)\,x\,m\,(x)$ for every x in N. Clearly then, $xf\,(x)\,x = x\,m\,(x)\,xm\,(x)\,x = xm\,(x)\,x = x$ and hence f is a mate

function for N. Also $f(x)\, x\, f(x) = m(x)\, xm(x)\, xm(x)\, xm(x) = m(x)\, xm(x) = f(x)$ and hence f is a mutual mate function for N.

Remark : If m is a mutual mate function for N, then apart from the results (ii) and (iii) of Lemma 3.2, we have $m(a)\, N = m(a)\, aN$ and $N\, m(a) = N\, a\, m(a)$ for every a in N.

Theorem 3.7—Let N be a nil near ring with a mate function m and let $g : N \to N$ be such that $g(x) = m(x)\, [xm(x) \pm x^{k-1}]$ for every x in N, where k is some definite integer > 1 such that $x^k = 0$ (k depending upon x). Then g is a mate function for N. If m is a mutual mate function for N, so is g.

Proof : Using the facts that $x^k = 0$ and m is a mate function for N, it is easy to get, from straight forward calculations, that $g(x)\, x = m(x)\, x$ and hence $x\, g(x)\, x = xm(x)\, x = x$. Thus g is a mate function for N. When m is a mutual mate function for N, we have $g(x)\, xg(x) = m(x)\, x\, g(x) = g(x)$ since $m(x)\, xm(x) = m(x)$ for all x in N. Hence the result follows.

Remark 3.7.1 : If N is an arbitrary near ring with a mutual mate function m and if $x^2 = 0$ for some x in N, then the element $m(x)\, [x\, m(x) + x]$ is a mutual mate of x.

For our further discussion throughout the rest of the paper, we assume that N has a mate function m.

4. Properties of Stable Rings

In this section we discuss the properties of a Stable near ring :

Theorem 4.1—N is Stable iff $E \subseteq C(N)$.

Proof : The necessity part follows from Lemma 2.4. For the sufficiency part, we observe that for all x in N, $Nx = N\, m(x)\, x = m(x)\, x\, N$ and hence $x\, N\, x = x\, m(x)\, x\, N = xN$. Similarly we get $xNx = Nx$ and the desired result follows.

Lemma 4.2—N has unique mutual mate function iff $E \subseteq C(E)$.

Proof : For the 'only if' part, we suppose that f is the unique mutual mate function for N. Clearly then, f is involutory as both x and $f(f(x))$ can serve as mutual mates of $f(x)$ for all x in N. Also f fixes every element of E. It is clear that for every x, y in E, both $y\, f(xy)$ and $f(xy)\, x$ serve as mutual mates of xy. The uniqueness of f demands that these mutual mates must be identical with $f(xy)$. It is easy to observe that $f(xy) \in E$ and hence $xy\ (= f(f(xy)) = f(xy)) \in E$. Thus $(E, .)$ is a sub semigroup of $(N, .)$. Clearly $f(yx)\ (= yx)$ also can serve as a mutual mate of xy. Hence, again from the uniqueness of f we get $xy = f(xy) = f(yx) = yx$. Hence $E \subseteq C(E)$. For the 'if' part, Lemma 3.6 guarantees the existence of mutual mate function f for N. If g is another mutual mate function for N, then for all x in N, $g(x)$

$= g(x)\, x\, g(x) = g(x)\, x f(x)\, x\, g(x) = f(x)\, x\, g(x)\, x\, g(x) = f(x)\, x\, g(x)$ $= f(x)\, x f(x)\, x\, g(x) = f(x)\, x g(x)\, x f(x) = f(x)\, x f(x) = f(x)$ and hence f is unique.

Theorem 4.3—If N has unique mutual mate function f then (i) N is zero-symmetric. (ii) f has the reversal law i. e. $f(x_1 \ldots\ldots x_k) = f(x_k).\ f(x_{k-1}) \ldots f(x_1)$ where x_i's $\in N$. (iii) $f(x^k) = (f(x))^k$ for every positive integer k and for every x in N. (iv) $L = \{0\}$. (v) N has IEP (vi) If $e \in E$ and $x \in N$ are such that $exe = xe$, then $e \in C(x)$. (vii) $E \subseteq C(N)$.

PROOF : (i) For every n in N, define $f_n : N \to N$ such that f_n agrees with f in N^* and $f_n(0) = n0$. Obviously f_n is a mutual mate function for N and hence $f_n = f$. Since f fixes every idempotent, the desired result follows.

(ii) We prove the reversal law for f by induction on 'k', the number of elements. When $k = 1$, the result holds trivially. We assume that the reversal law holds good for any set of k elements of N. Let $x_1, \ldots, x_k \in N$ and for convenience let $x = x_1 . x_2 . \ldots . x_k$. Let y be any element of N. To get the desired result by simple induction, we need only to prove that $f(xy) = f(y)\, f(x)$. Clearly $xy = (xf(x)\, x)(y f(y)\, y) = x\, (f(x)\, x)(y f(y))\, y = x\, y f(y) f(x)\, x\, y$ (using Lemma 4.2). In the same vein we can prove that $f(y) f(x) = f(y) f(x)\, xy\, f(y) f(x)$. Hence $f(y)\, f(x)$ is a mutual mate of xy. As $f(xy)$ is the unique mutual mate of xy, we get $f(xy) = f(y) f(x)$.

(iii) follows by taking $x = x_1 = x_2 = \ldots = x_k$ in (ii).

(iv) Suppose $x^2 = 0$ for some x in N, we need only to prove that $x = 0$ (Prob. 14, p. 9 of McCoy[3], valid for N also). Since $x^2 = 0$, we have $0 = f(x^2) = (f(x))^2$ (using (iii)). Clearly then, we have $f(x) = f(x)\, [x f(x) + x]$ (by Remark 3.7.1 and the fact that the R.H.S. is also a mutual mate of x). Hence $0 = f(x^2) = (f(x))^2 = f(x)\, [x f(x) + x]\, f(x) = f(x)\, [0 + x f(x)] = f(x)\, x f(x) = f(x)$. The uniqueness of f forces '$x = 0$'.

(v) From (i) and (iv), $N = N_0(0)$ and hence N has IFP by Lemma ?.2

(vi) $exe = xe \Rightarrow (ex - xe)\, e = 0 \Rightarrow e\,(ex - xe) = 0 \Rightarrow ex\,(ex - xe) = 0$ (by IFP). Also $xe\,(ex - xe) = x0 = 0$. Hence $(ex - xe)^2 = 0$ and the result follows as $L = \{0\}$.

(vii) We have for every e in E and for every x in N, $(x f(x)\, e - x f(x))\, e = 0$. Hence by Lemma 4.2, $(ex f(x) - x f(x))\, e = 0$. By IFP, $(e\, x f(x) - x f(x))\, xe = 0$. Hence $e\, x f(x)\, xe = x f(x)\, x\, e$ i. e. $e\, x\, e = x\, e$ and (vi) takes care of the rest of the proof.

Theorem 4.1, Lemma 4.2 and part (vii) of Theorem 4.3 guarantee the following result :

Theorem 4.4—N is Stable iff it has unique mutual mate function.

In Remark 2.1.4, we have observed that the Subcommutativity of N does not imply its Stability in general. But the fact that N admits a mate function readily guarantees the following :

Theorem 4.5—N is Stable iff it is Subcommutative.

PROOF : The necessity part is obvious. For the sufficiency part, we observe that for every x in N, $xN = Nx = Nxm(x)x = xN\,m(x)\,x = x\,Nx$. Hence the result.

Theorem 4.6—The following statements are equivalent :
(i) N is a nearfield (ii) N is Stable and subdirectly irreducible (iii) N is Stable and none of the non zero idempotents is a zero divisor,

PROOF : "(ii) $\Rightarrow$ (ii)" is obvious.

To prove "(ii) $\Rightarrow$ (iii)", let E_1 be the set of all elements of E^* which are zero divisors. Suppose E_1 is not empty. Let I be the intersection of all the annihilator ideals of elements in E_1. Clearly then $I \neq \{0\}$, as N is subdirectly irreducible (by 1.60, p. 25 of Pilz[4]). If $n \in I^*$, then $m(n)\,n \in E_1 \cap I^*$. This immediately leads to the obvious contradiction and (iii) follows.

To prove '(iii) $\Rightarrow$ (i)' we observe that for every e in E^*, $Ne = Ne^2$ and hence $N = Ne\ (= eNe = eN)$. It follows that E^* consists of a single element e say, which is the two sided identity of $(N, .)$ Hence for every x in N^*, $m(x)$ serves as the inverse of x and (i) follows.

Remark 4.6.1: Several authors have discussed different necessary and sufficient conditions for a near ring to be a near field—a list, through not complete in itself, can be found in (8. 3, p. 237 of Pilz[4]). Theorem 3 of Beidleman[1] furnishes one such condition for a regular near ring. Beidleman assumes the existence of the two sided identity.

Corollary 4.7—If N is Stable, it is isomorphic to a subdirect sum of near fields.

PROOF : From 1.62, p. 26 of Pilz[4], N is isomorphic to a subdirect sum of subdirectly irreducible near rings N_i 's say—each of which is a homomorphic image of N (by 1.58 Remarks, p. 25 of Pilz[4]). Obviously, the defining properties of Stability and the existence of a mate function are preserved under homomorphisms. Hence each N_i is Stable and admits a mate function. Theorem 4.6 takes care of the rest of the proof.

5. PROPERTIES OF PSEUDO STABLE NEAR RINGS

In this section, we discuss the properties of pseudo stable near rings:

Lemma 5.1—If N has (PS), $Nx\,m(x) = N\,m(x)\,x$ for all x in N.

PROOF : For all x in N, we have, $xN = x\,m(x)\,N$ and hence $Nx = Nx\,m(x)$. i. e. $N\,m(x)\,x = Nx\,m(x)$

Theorem 5.2—If N has (PS), then for every a in N, there exists some a' in N such that (i) $a = a'\ m\ (a)\ a^2$ (ii) $f : N \to N$, with $f(a) = a'\ m\ (a)$, is a mate function for N. and (iii) $f(a) \in C(a)$.

Proof : (i) From Lemma 5.1 we have $Na\ m\ (a) = N\ m\ (a)\ a$ for all a in N. For $a\ m\ (a)$ in N, there exists some a' in N, such that $a\ m\ (a)\ a\ m\ (a) = a'\ m\ (a)\ a$ i.e. $a\ m\ (a) = a'\ m\ (a)\ a$ and hence $a = a\ m\ (a)\ a = a'\ m\ (a)\ a^2$.

(ii) Let $b = a\ f\ (a)\ a$. Obviously $ba = a\ f\ (a)\ a^2 = aa = a^2$ (by part (i)). Also $b^2 = a\ f\ (a)\ a^2\ f\ (a)\ a = a\ a\ f\ (a)\ a = a\ b$. These facts together with part (i) gurantee that the conditions of Lemma 2.5 are satisfied. Hence $a\ (=b) = a\ f\ (a)\ a$ and (ii) follows.

(iii) Let $x = a\ f\ (a)$ and $y = f\ (a)\ a$. Let us take $x - y = w_1, aw_1, aw_1 = w_2$ and $xw_1 = w_3$. From (i) and (ii) we observe thet $w_1\ a = 0 = w_1\ x$. Closely following the pattern of proof of Lemma 2.5 we get $w_2 = w_2^2 = a0$ and $xw_1\ (= w_3 = w_3^2) = x\ 0$. Since $yw_1 = f\ (a)\ w_2$, we have $yw_1 = f\ (a)\ a\ 0 = y\ 0$. Thus $xw_1 - yw_1 = (x - y)\ 0$. i. e. $w_1^2 = w_1\ 0$. It is now easy to observe that $w_1\ (= w_1^2 = w_1\ 0) = w_1\ a = 0$ and the desired result follows

Corollary 5.3—If N has (PS), it has a mutual mate function g which is unique with the property that $g\ (a) \in C\ (a)$.

Proof : By Theorem 5.2, N has a mate function f with $f\ (a) \in C\ (a)$. We set $g : N \to N$ such that $g\ (a) = f\ (a)\ a\ f\ (a)$ for all a in N.

By Lemma 3.6, g is a mutual mate function for N. Also for all a in N, $a\ g\ (a) = a\ f\ (a)\ a\ f\ (a) = f\ (a)\ a\ f\ (a)\ a = g\ (a)\ a$. Hence g has the desired property. Suppose h is another mutual mate function with the same property. For all a in N,$h\ (a) = h\ (a)\ a\ h\ (a) = a\ h\ (a)\ h\ (a) = a\ g\ (a)\ a\ h\ (a)\ h\ (a) = g\ (a)\ a\ h\ (a)\ a\ h\ (a) = g\ (a)\ ah\ (a) = g\ (a)\ ag\ (a)\ h\ (a)\ a = g\ (a)\ g\ (a)\ a\ h\ (a)\ a = g\ (a)\ g\ (a)\ a = g\ (a)\ a\ g\ (a) = g\ (a)$ and the desired result follows.

Corollary 5.4—If N has (PS) and is zero-symmetric, N has IFP.

Proof : Following the notations of Theorem 5.2, we observe that $a = a\ f\ (a)\ a = a^2\ f\ (a)$. Since $a^2 = 0 \Rightarrow a = 0$ we have $L = \{0\}$. Thus $N = N_0\ (0)$ and the desired result follows from Lemma 2.3

Theorem 5.5—If N has (PS), $xN = xNx$ for all x in N.

Proof : For e in E and n in N, let $a = en$ and $b = ene$. Clearly then, $ab = (en)\ (ene) = ene\ ene = b^2$ and $ba = (ene)\ (en) = en\ en = a^2$. These facts together with part (i) of Theorem 5.2 guarantee that the conditions of Lemma 2.5 are satisfied. Hence $(a =)\ en = ene\ (= b)$ for all e in E and for all n in N. It follows easily that $e\ N = eNe$ for all e in E. Thus for all x in N, $xN = x\ m\ (x)\ xN = x\ (m\ (x)\ x\ N) = x\ (m\ (x)\ x\ N\ m\ (x)\ x) = (x\ m\ (x) x)\ Nm\ (x)\ x = x\ Nx$.

Theorem 5.6—The following statements are equivalent:

(i) N is Stable. (ii) N has (PS), $N = N_0$ and $E \subseteq N_d$. (iii) N has (PS) and $ne = ene$ for all n in N and for all e in E.

PROOF : '(i) $\Rightarrow$ (ii)' is obvious.

To prove '(ii) $\Rightarrow$ (iii)', we first observe that $N = N_0\,(0)$. Hence $e\,(ne - ene) = 0 \Rightarrow (ne - ene)\,e = 0$ i. e. $ne = ene$ for all e in E and for all n in N.

To prove '(iii) $\Rightarrow$ (i)', we need only to appeal to Theorem 5.5, we have then, $en\,(=ene) = ne$. Thus $E \subseteq C\,(N)$ and (i) follows from Theorem 4.1.

Thearem 5.7—If N has (PS) and $N = N_0$, then N has a mutual mate function g such that for all x in N_d, $g\,(x)\,x \in C\,(N)$.

PROOF : The mutual mate function 'g' introduced in Corollary 5.3 comes in handy to serve the purpose. For every n in N and for every x in N_d, we have,

$$xng\,(x)\,x = x\,g\,(x)\,xng\,(x)\,x = x\,(g\,(x)\,x\,n\,g\,(x)\,x)$$

$$= x\,g\,(x)\,x\,n \text{ (from the proof of Theorem 5.5).}$$

Hence $x\,(n\,g\,(x)\,x - g\,(x)\,x\,n) = 0$ and since $N = N_0\,(0)$ we have $(n\,g\,(x)\,x - g\,(x)\,x\,n)\,x = 0$. By IFP we have, $(n\,g\,(x)\,x - g\,(x)\,x\,n)\;g\,(x)\,x = 0$. Thus $n\,g\,(x)\,x = g\,(x)\,x\,n\,g\,(x)\,x = g\,(x)\,x\,n$. and the desired result follows.

Theorem 5.8—When N is a ring, it is Stable iff it has (PS).

PROOF : If g is the mutual mate function of Theorem 5.7, we first observe that $g\,(e) = e$ for all e in E and since $E \subseteq N = N_d$, we get $e = e^2 = g\,(e)\,e \in C\,(N)$. Hence N is Stable by Theorem 4.1. The 'only if' part is obvious.

Theorem 5.9 – Let N be zero-symmetric. Then N is a near field iff it has (RPS) and none of its non zero idempotents is a zero divisor.

PROOF : For every e in E^*, we have $Ne^2 = Ne$ and hence $Ne = N$. This guarantees that every e in E^* is a right identity. Since $Nx\;(= N) = Ny$ for all x, y in E^* and since N has (RPS), we have $xN = yN$. Hence $xy = ys$ for some s in N. i. e. $x = ys$. Hence $y = yx = y^2 s = ys = x$ and consequently E^* consists of only one element, e say. For every n in N^*, we must have $n\,m\,(n) = e = m\,(n)\,n$. It follows that e is the two sided identity of $(N, .)$ and $m\,(n)$ is the inverse of n. The converse is obvious.

Theorem 5.10 – Let $N = N_0$ be Pseudo Stable. Then N is a near field iff none of the non zero idempotents is a zero divisor and atleast one of them is in N_d.

PROOF : As in the proof of Theorem 5.9, every e in E^* is a right identity. Let $d \in E^* \cap N_d$. We observe that for all e in E^*, $d\,(e - d) = 0$ and since $N = N_0\,(0)$ (from the proof of Corollary 5.4), we have $(e - d)\,d = 0$. Hence $e = d$. Clearly then, $E^* = \{d\}$. The rest of the proof is exactly as in the proof of Theorem 5.9.

We conclude our discussion with the following:

Theorem 5.11—N is a near field iff all possible mate functions of N agree in N^*.

PROOF: The 'only if' part is obvious. For the 'if' part, we first observe that every element of N is a mate of $x0$ for all x in N and as such $x0 \notin N^*$. Hence $N = N_0$. Also for every e in E^* and for every mate function m of N, $m(e) = e$. If $xe = 0$ for some x in N, it is clear that both e and $x + m(e)$ serve as mates of e. This forces $x = 0$ and as such none of the non-zero idempotents is a right zero divisor. It follows that every e in E^* is a right identity. For every x in N^* both $x\,m(x)$ and $m(x)x$ are in E^* and serve as mates of e. Thus we have $x\,m(x) = e = m(x)x$ and hence $e\,x = x = xe$. (Also $e\,0 = 0 = 0\,e$ as $N = N_0$). These facts force $E^* = \{e\}$ where e is the two sided identity of $(N, .)$ and the desired result follows.

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DEGREE OF L_1-APPROXIMATION TO INTEGRABLE FUNCTIONS BY BERNSTEIN TYPE OPERATORS

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This paper deals with the degree of L_1-approximation to integrable functions by integrated Meyer-König and Zeller operators in terms of the L_1-modulus of continuity.

1. INTRODUCTION AND RESULTS

It is well known that the nth operator M_n, $n \in N$, of Meyer-König and Zeller is associating with a bounded function $f : I = [0, 1] \rightarrow R$ the so called nth Bernstein power series

$$M_n (f, x) = \sum_{k=0}^{\infty} m_{n,k} (x) f \left(\frac{k}{k+n} \right) \qquad ..(1.1)$$

where

$$m_{n,k} (x) = \binom{k+n}{k} (1-x)^{n+1} x^k$$

converging for $0 \leqslant x < 1$. Meyer-König and Zeller[4] proved that the sequence $(M_n)_{n \in N}$ gives a linear approximation method on the normed space $(C (I), \| \cdot \|_\infty)$ (with $\| \cdot \|_\infty$ the usual supnorm on I), i.e. $\lim_{n \to \infty} \| f - M_n f \|_\infty = 0$ for all $f \in C (I)$. Its degree of approximation can be estimated by Lupas and Müller[2].

$$\| f - M_n f \|_\infty \leqslant \frac{31}{27} w_{1,\infty} \left(f, \frac{1}{\sqrt{n}} \right) (n \in N)$$

where $w_{1,\infty} (f, \cdot)$ is ordinary modulus of continuity of f with respect to the sup-norm.

A small modification of Meyer-König and Zeller operators due to Müller makes it possible to approximate Lebesgue integrable functions in the L_1 norm by the integrated Meyer-König and Zeller operators.

$$\hat{M}_n(f, x) = \sum_{k=0}^{\infty} \hat{m}_{n,k}(x) \int_{\frac{k}{k+n}}^{\frac{k+1}{k+n+1}} f(t)\, dt$$

where

$$\hat{m}_{n,k}(x) = (n+1) \binom{k+n+1}{k} (1-x)^n x^k.$$

The L_1 analog of Meyer-König and Zeller's result was established by Müller[6] who has proved that for every Lebesgue integrable function f on [0, 1],

$$\int_0^1 | \hat{M}_n(f, x) - f(x) | \, dx \to 0 \ (n \to \infty).$$

As far as estimates of the degree of approximation to Lebesgue integrable functions by the operators $\hat{M}_n(f)$ in the L_1 norm are concerned, very little is known. A result which gives the degree of approximation of f by some Bernstein type operators for a very special class of Lebesgue integrable functions f is due to Leviatan[3]. Leviatan's result may be stated in our notation as follows :

If f is a Lebesgue integrable function on [0, 1], of bounded variation on every closed subinterval of (0, 1), then

$$\int_0^1 | \hat{M}_n(f, x) - f(x) | \, dx < (2/e)^{1/2} J(f)\, n^{-1/2}$$

where

$$J(f) = \int_0^1 \sqrt{x(1-x)}\, | df(x) | .$$

This result is useful when $J(f) < \infty$.

In this paper we shall show that

$$\int_0^1 \sqrt{x(1-x)}\, | \hat{M}_n(f, x) - f(x) | \, dx$$

can be estimated in terms of the L_1 modulus of continuity

$$w_{L_1}(f, \delta) = \sup \{ \int_0^1 | f(x+t) - f(x) | \, dx : | t | \leqslant \delta \}.$$

We assume here and in the rest of the paper that the function f is extended to $(-\infty, \infty)$ by periodicity with period 1 (its value at the integers is immaterial). The L_1 norm with weight function $w(x) = \sqrt{(1-x)}$ seems to be a more convenient norm

than the usual L_1 norm for the study of approximation properties of integrated Meyer-König and operators.

Our result may be stated as follows.

Theorem 1—Let f be a Lebesgue function on [0, 1]. Then, for $n \geqslant 2$,

$$\int_0^1 \sqrt{x(1-x)}\,|\,\hat{M}_n(f, x) - f(x)\,|\,dx < \frac{2\pi^2}{3} w_{L_1}(f, n^{-1/2}) + O(n^{-2}) \qquad \ldots(1.2)$$

where

$$w_{L_1}(f, \delta) = \sup\{\int_0^1 |\,f(x+t) - f(x)\,|\,dx : |\,t\,| \leqslant \delta\}.$$

The proof will be tailored specially for the case of the L_1 norm and follows ideas in a paper by Bojanic and Shisha[1].

2. Lemmas

The proof of our theorem is based on two lemmas.

Lemma 1—If f is a Lebesgue integrable function on [0, 1], then, for $n \geqslant 2$ ($n \in N$) and $x, t \in [0, 1]$, we have

$$x(1-x)^2(\hat{M}_{n-1}(f, x) - f(x))$$

$$\leqslant \sum_{k=0}^{\infty} n\, m_{n,k}(x)\left(\frac{k}{n+k} - x\right) \int_0^{\frac{k}{n+k+1} - x} (f(x+t) - f(x))\,dt.$$

Proof : We have

$$\hat{M}_{n-1}(f, x) \int_0^1 = K_n(x, t) f(t)\,dt$$

where

$$K_n(x, t) = \sum_{k=0}^{\infty} \hat{m}_{n-1,k}(x)\, \chi_{\left[\frac{k}{n+k-1}, \frac{(k+1)}{n+k}\right]}(t)$$

$\chi_{\left[\frac{k}{k+n-1}, \frac{k+1}{k+n}\right]}(t)$ being the characteristic function of $\left[\frac{k}{k+n-1}, \frac{(k+1)}{k+n}\right]$.

By partial summation we find for $s \in N$, $n \geqslant 2$ and $x, t \in [0, 1]$ that

$$\sum_{k=0}^{s} \hat{m}_{n-1,k}(x)\, \chi_{\left[\frac{k}{k+n-1}, \frac{k+1}{k+n}\right]}(t)$$

(*equation continued on p.* 1220)

$$= \sum_{k=1}^{s} (\hat{m}_{n-1,\, k-1}(x) - \hat{m}_{n-1,k}(x))\, \chi_{\left[0,\, \frac{k}{k+n-1}\right]}(t)$$

$$+ \hat{m}_{n-1,s}(x)\, \chi_{\left[0,\, \frac{s+1}{n+k-1}\right]}(t).$$

Since

$$\begin{aligned}\hat{m}_{n-1,\, k-1}(x) - \hat{m}_{n-1,k}(x) &= n \binom{k+n-1}{k-1} x^{k-1}(1-x)^{n-1} \\ &\quad - n\binom{k+n}{k} x^{k}(1-x)^{n-1} \\ &= n\left[\binom{n+k-1}{k-1} - \binom{n+k}{k} x\right] \\ &\quad \times x^{k-1}(1-x)^{n-1} \\ &= n\binom{n+k}{k}\left[\frac{k}{k+n} - x\right] x^{k-1}(1-x)^{n-1}\end{aligned}$$

and

$$\lim_{s\to\infty} \hat{m}_{n-1,s}(x) \to 0$$

we have

$$\begin{aligned}&x(1-x)^2 (\hat{m}_{n-1,\, k-1}(x) - \hat{m}_{n-1,k}(x)) \\ &\qquad = n\binom{n+k}{k} x^k (1-x)^{n+1}\left(\frac{k}{n+k} - x\right) \\ &\qquad = n\, m_{n,k}(x)\left(\frac{k}{n+k} - x\right).\end{aligned}$$

Now it follows that

$$x(1-x)^2 K_n(x, t)$$

$$= \sum_{k=0}^{\infty} n\, m_{n,k}(x)\left[\frac{k}{n+k} - x\right] \chi_{\left[0,\, \frac{k}{n+k-1}\right]}(t).$$

Hence

$$x(1-x)^2 \hat{M}_{n-1}(f, x)$$

$$= \sum_{k=0}^{\infty} n\, m_{n,k}(x)\left(\frac{k}{n+k} - x\right) \int_0^{\frac{k}{n+k-1}} f(t)\, dt$$

(*equation continued on p.* 1221)

$$= \sum_{k=0}^{\infty} n\, m_{n,k}(x)\left(\frac{k}{n+k} - x\right) \int_{x}^{\frac{k}{n+k-1}} f(t)\, dt.$$

Thus, the proof the lemma is complete, since

$$\int_{0}^{\frac{k}{n+k-1}} f(t)\, dt = \int_{0}^{x} f(t)\, dt + \int_{x}^{\frac{k}{n+k-1}} f(t)\, dt$$

$$= \int_{0}^{x} f(t)\, dt + \int_{0}^{\frac{k}{n+k-1} - x} f(x+t)\, dt$$

and

$$\sum_{k=0}^{\infty} \left(\frac{k}{n+k} - x\right)^2 m_{n,k}(x) = x(1-x)^2/n + \frac{x(1-x)^2(2x-1)}{n^2}$$

$$+ O(n^{-3}).$$

Our second lemma is a more precise version of the known inequalities [see Sikkema[8] (431-435), Müller[7]].

Lemma 2—For $n \geqslant 2$ and $x \in [0, 1]$ we have

$$\sum_{k=0}^{\infty} \left|\frac{k}{n+k} - x\right|^5 m_{n,k}(x) < x(1-x)^2/n^{5/2}\, O(n^{-3}).$$

PROOF : We have

$$\sum_{k=0}^{\infty} \left|\frac{k}{n+k} - x\right|^5 m_{n,k}(x) \leqslant \left(\sum_{k=0}^{\infty} \left(\frac{k}{n+k} - x\right)^4 m_{n,k}(x)\right)^{1/2}$$

$$\times \left(\sum_{k=0}^{\infty} \left(\frac{k}{n+k} - x\right)^6 m_{n,k}(x)\right)^{1/2}$$

and the result follows, since

$$\sum_{k=0}^{\infty}\left(\frac{k}{n+k} - x\right)^4 m_{n,k}(x) = \sum_{k=0}^{\infty}\left(\frac{k}{n+k}\right)^4 m_{n,k}(x)$$

(*equation continued on p.* 1222)

$$
\begin{aligned}
&- \sum_{k=0}^{\infty} 4x \left(\frac{k}{n+k} \right)^3 \\
&\times m_{n,k}(x) + \sum_{k=0}^{\infty} 6x^2 \left(\frac{k}{n+k} \right)^2 m_{n,}(x) - 3x^4 \\
= x\Bigg[& x^3 + \frac{3x^2(1-x)^2}{n} \\
&+ \frac{x(1-x)^2(1-2x+11x^2)}{n^2} + \frac{3x^3(1-x)^2}{n} \\
&+ \frac{3x^3(1-x)^2}{n^2} - \frac{21x^3(1-x)^3}{n^3} \\
&+ \frac{3x^2(1-x)^4}{n^2} + \frac{3x^2(1-x)^3}{n^2} \Bigg] \\
&- 4x\left[x^3 + \frac{3x^2(1-x)^2}{n} \right. \\
&\left. + \frac{x(1-x)^2(1-2x+11x^2)}{n^2} \right] \\
&+ 6x^2\left[x^2 + \frac{x(1-x)^2}{n} \right. \\
&\left. + \frac{x(1-x)^2(2x-1)}{n^2} \right] - 3x^4 + O(n^{-3}) \\
= & \frac{3x^2(1-x)^4}{n^2} + O(n^{-3}) \\
&< \frac{x(1-x)^2}{n^2} + O(n^{-3})
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{k=0}^{\infty} \left(\frac{k}{n+k} - x \right)^6 m_{n,k}(x) = & \sum_{k=0}^{\infty} \left(\frac{k}{n+k} \right)^6 m_{n,k}(x) \\
&- \sum_{k=0}^{\infty} 6x \left(\frac{k}{n+k} \right)^5 m_{n,k}(x) \\
&+ \sum_{k=0}^{\infty} 15x^2 \left(\frac{k}{n+k} \right)^4
\end{aligned}
$$

(*equation continued on p. 1223*)

$$\times m_{n,k}(x) - \sum_{k=0}^{\infty} 20x^3 \left(\frac{k}{n+k}\right)^3 m_{n,k}(x)$$

$$+ \sum_{k=0}^{\infty} 15x^4 \left(\frac{k}{n+k}\right)^2 m_{n,k}(x) - 5\,x^6$$

$$= \frac{5\,x^3(1-x)^6}{n^3} + O(n^{-4})$$

$$< \frac{x(1-x)^2}{n^3} + O(n^{-4})$$

for $x \in [0, 1]$.

3. Proof of the Theorem

Let $x \in (0, 1)$. By Lemma 1 we have

$$x(1-x^2)\,|\,\hat{M}_{n-1}(f, x) - f(x)\,|$$

$$\leqslant \sum_{k=0}^{\infty} n\, m_{n,k}(x) \left|\frac{k}{n+k} - x\right| \left| \int_0^{\frac{k}{n+k-1} - x} (f(x+t) - f(x))\, dt \right|$$

$$\leqslant \sum_{k=0}^{\infty} n\, m_{n,k}(x) \left|\frac{k}{n+k} - x\right| \int_{-\left|\frac{k}{n+k-1} - x\right|}^{\left|\frac{k}{n+k-1} - x\right|} |\,f(x+t) - f(x)\,|\, dt$$

$$\leqslant \sum_{r=0}^{[1/\delta]} I_{n,r}(x)$$

where $\delta \in (0, 1)$ and

$$I_{n,r}(x) = \sum_{r\delta < \left|\frac{k}{n+k-1} - x\right| \leqslant (r+1)\delta} m_{n,k}(x) \left|\frac{k}{n+k} - x\right| \int_{-\left|\frac{k}{n+k-1} - x\right|}^{\left|\frac{k}{n+k-1} - x\right|} |\,f(x+t) - f(x)\,|\, dt.$$

Clearly

$$I_{n,r}(x) \leqslant S_r(n, \delta; x) \int_{-(r+1)\delta}^{(r+1)\delta} |\,f(x+t) - f(x)\,|\, dt$$

where

$$S_r(n, \delta; x) = \sum_{r\delta < \left|\frac{k}{n+k-1} - x\right| \leq (r+1)\delta} n\, m_{n,k}(x) \left|\frac{k}{n+k} - x\right|.$$

Hence, it follows that

$$x(1-x)^2 \left| \hat{M}_{n-1}(f, x) - f(x) \right| \leqslant \sum_{r=0}^{[1/\delta]} S_r(n, \delta; x)$$

$$\int_{-(r+1)\delta}^{(r+1)\delta} | f(x+t) - f(x) | \, dt. \qquad \ldots(3.1)$$

Next we shall estimate the coefficient $S_r(n, \delta; x)$ for $r = 0$ and $1 \leqslant r \leqslant [1/\delta]$. We have first

$$S_0(n, \delta; x) = \sum_{\left|\frac{k}{n+k-1} - x\right| \leqslant \delta} n\, m_{n,k}(x) \left|\frac{k}{n+k} - x\right|$$

$$\leqslant \sum_{k=0}^{\infty} n\, m_{n,k}(x) \left|\frac{k}{n+k} - x\right|$$

$$< n^{1/2} \sqrt{x}(1-x) + O(n^{-2}). \qquad \ldots(3.2)$$

Next, for $1 \leqslant r \leqslant [1/\delta]$, we have, by Lemma 2

$$S_r(n, \delta; x) \leqslant n(r+1)^{-4} \delta^{-4} \sum_{r\delta < \left|\frac{k}{n+k-1} - x\right| \leqslant (r+1)\delta}$$

$$\left|\frac{k}{n+k} - x\right|^5 m_{n,k}(x)$$

$$\leqslant n(r+1)^{-4} \delta^{-4} \sum_{k=0}^{\infty} \left|\frac{k}{n+k} - x\right|^5 m_{n,k}(x)$$

$$< n^{-3/2} x(1-x)^2 (r+1)^{-4} \delta^{-4} + O(n^{-3}). \qquad \ldots(3.3)$$

From (3.1), (3.2) and (3.3) it follows that

$$\sqrt{x}(1-x) \left| \hat{M}_{n-1}(f, x) - f(x) \right|$$

$$< n^{1/2} \int_{-\delta}^{\delta} | f(x+t) - f(x) | \, dt + 1/2\, n^{-3/2} \delta^{-4}$$

$$\times \sum_{r=1}^{[1/\delta]} (r+1)^{-4} \int_{-(r+1)\delta}^{(r+1)\delta} | f(x+t) - f(x) | \, dt$$

$$+ O(n^{-2}).$$

Integrating this inequality and taking into account that

$$\int_{-(r+1)\delta}^{(r+1)\delta} \left(\int_0^1 | f(x+t) - f(x) | \, dx\right) dt \leqslant 2(r+1)\delta \, w_{L_1}(f, (r+1)\delta)$$

we find that

$$\int_0^1 \sqrt{x(1-x)} \, | \hat{M}_{n-1}(f, x) - f(x) | \, dx$$

$$< 2n^{1/2} \delta \, w_{L_1}(f, \delta) + n^{-3/2} \delta^{-3} \sum_{r=1}^{[1/\delta]} (r+1)^{-3} w_{L_1}(f, (r+1)\delta) \,|$$

$$+ O(n^{-2}).$$

Choosing here $\delta = n^{-1/2}$, we find that

$$\int_0^1 \sqrt{x(1-x)} \, | \hat{M}_{n-1}(f, x) - f(x) | \, dx$$

$$< 2w_{L_1}(f, n^{-1/2}) + \sum_{r=0}^{[1/n^{-1/2}]} (r+1)^{-3} w_{L_1}(f, (r+1)/n^{1/2})$$

$$\leqslant 2 \sum_{k=1}^{[1/n^{-1/2}]+1} k^{-3} w_{L_1}(f, k/n^{1/2}) + O(n^{-2}).$$

Since the L_1 modulus of continuity is a subadditive function, we have, for every $0 < h_1 \leqslant h_2$.

$$\frac{2w_{L_1}(f, h_1)}{h_1} \geqslant \frac{w_{L_1}(f, h_2)}{h_2}$$

(see Timan[10], p. 112). In particular we have, for $k \geqslant 1$,

$$w_{L_1}(f, k/n^{1/2}) \leqslant 2\, k \, w_{L_1}(f, n^{-1/2}).$$

Hence

$$\int_0^1 \sqrt{x(1-x)} \, | \hat{M}_{n-1}(f, x) - f(x) | \, dx$$

$$< 4\, w_{L_1}(f, n^{-1/2}) \sum_{k=1}^{\infty} k^{-2} + O(n^{-2})$$

$$\leqslant \frac{2\pi^2}{3} w_{L_1}(f, n^{-1/2}) + O(n^{-2}) \qquad \ldots(3.4)$$

and the theorem is proved.

Since the expression (1.2) for integrated Meyer-König and Zeller operator is a strict inequality for all n while in the case of Bernstein-Kantorovitch operator the equality may also hold for some n, the approximation of functions given by integrated

Meyer-König and Zeller operator for some n is better than that given by Bernstein-Kantorovitch operator.

Remark : The degree of approximation for operatar $\hat{M}_n(f; x)$ cannot be improved further. For the value of the constant c for which

$$\int_0^1 \sqrt{x(1-x)}\, |\hat{M}_n(f; x) - f(x)|\, dx < c\, w_{L_1}(f, n^{-1/2}) + O(n^{-2}) \qquad \ldots(3.5)$$

is always less than $2\pi^2/3$. Moreover $c \geqslant 1$, which can be seen from the following example. Let $\delta_n = O\sqrt{(1/n)}$ and suppose that $f_n(x)$ is the function which is equal to zero at x_0, $0 < x_0 < 1$, equal to 1 in $[\theta, x_0 - \delta_n]$ and $[x_0 + \delta_n, 1]$ and linear in the rest of $[0, 1]$. For large n, we have $w_{L_1}(\delta_n) = 1$ for f_n ; also

$$|\hat{M}_n(f; x) - f_n(x_0)| = \hat{M}_n(f; x_0) \geqslant \sum_{\left|\frac{k}{(k+n-1)} - x_0\right|} m_{n-1,k}(x_0)$$

$$= 1 - \epsilon_n. \qquad \ldots(3.6)$$

Therefore (3.5) can not be true if $c < 1$.

The function $w_{L_1}(\delta)$ can not, therefore, be replaced in (3.4) by any other function decreasing to zero more rapidly.

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TRANSIENT MAGNETOTHERMOELASTIC WAVES IN A HALF-SPACE WITH THERMAL RELAXATIONS

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The distribution of temperature, deformation and magnetic field in a homogeneous isotropic, thermally and perfectly electrically conducting, elastic half-space, in contact with the vacuum, has been investigated by taking (i) a step in stress and (ii) a thermal shock at the plane boundary, in the context of Green-Lindsay theory of thermoelasticity. The Laplace transform on time has been used to obtain the solutions. Because the "second sound" effects are short-lived, so the small time approximations have been considered. The deformation and temperature are found to be continuous at the wavefronts whereas the magnetic field is found to be discontinuous in the case of normal load. But these quantities are discontinuous in the case of thermal shock.

1. INTRODUCTION

The magnetothermoelastic disturbances in a perfectly conducting elastic half-space, in contact with vacuum, due to an applied thermal disturbance on the plane boundary were studied by Kaliski and Nowacki[1] in the absence of coupling between temperature and strain fields. The problem[1] was also considered by Massalas and Dalmangas[2] by taking into account thermomechanical couplings. The problem[2] was then extended to generalized thermoelasticity theory developed by Green and Lindsay[3] involving two relaxation times by Chatterjee and Roychoudhuri[4].

In the present paper the distributions of deformation, temperature, and perturbed magnetic field, from (i) a normal load and (ii) a thermal shock acting on the boundary of an half-space, are obtained by employing generalised theory of thermoelasticity developed by Green and Lindsay[3]. The Laplace transform[5] technique is used to obtain the solutions. As the "second sound" effects are short lived, so small time approximations have been considered.

2. THE PROBLEM AND ITS SOLUTION

We consider a homogeneous isotropic thermally conducting elastic medium at uniform temperature T_0, in contact with vacuum. We suppose that an initial magnetic field is acting along x_3-axis in both the media. The simplified linear equations of ele-

ctrodynamics of slowly moving bodies for a perfectly homogeneous conducting elastic medium are

$$\vec{\nabla} \times \vec{h} = \frac{4\pi}{c} \vec{J}, \quad \vec{\nabla} \times \vec{E} = -\frac{\mu_0}{c} \dot{\vec{h}}$$

$$\vec{\nabla} . \vec{h} = 0, \quad \vec{E} = \frac{-\mu_0}{c} (\dot{\vec{u}} \times \vec{H_0}) \qquad \ldots(1)$$

where $\vec{h}$ denotes the perturbation of magnetic field, $\vec{J}$ is the elastic current density vector, $\vec{E}$ the electricfield, $\vec{H_0}$ the initial magnetic field, $\vec{u}$ the displacement vector, μ_0 is the magnetic permeability, and c the velocity light. The superposed dot represen ts the differentiation with respect to time.

The equations of motion and heat conduction in the context of Green and Lindsay theory[3] of thermoelasticity are

$$\mu \nabla^2 \vec{u} + (\lambda + \mu) \vec{\nabla}\, \vec{\nabla}. \vec{u} + \frac{\mu_0}{4\pi} [(\vec{\nabla} \times \vec{h}) \times \vec{H_0}]$$

$$- \gamma (\vec{\nabla} \theta + \alpha \vec{\nabla} \dot{\theta}) = \rho \ddot{\vec{u}} \qquad \ldots(2)$$

and

$$\rho C_v (\dot{\theta} + \alpha^* \ddot{\theta}) + \gamma T_0 \dot{\Delta} = K \theta_{,ii} \, (i, j = 1, 2, 3). \qquad \ldots(3)$$

where λ, μ are the Lame' constants, $\gamma = (3\lambda + 2\mu) \alpha_T$, α_T the coefficient of linear thermal expansion, $\theta = T - T_0$, T the absolute temperature, T_0 the uniform temperature of the body in its natural state, $K = \lambda_T C_\varepsilon$, λ_T represents the coefficient of heat conduction, C_ε the specific heat at constant strain, ρ the mass density, C_v the specific heat at constant volume, and α, α^* are thermal relaxation times.

For $H_0 = (0, 0, \vec{H_3})$, eqns. (1) become

$$\vec{E} = \frac{\mu_0 H_3}{c} (0, \dot{u}_1, 0), \quad \vec{h} = -\frac{c}{\mu_0} \left(0, 0, \frac{\partial E_2}{\partial x_1}\right),$$

$$\vec{J} = \frac{c}{4\pi} \left(0, -\frac{\partial h_3}{\partial x_1}, 0\right), \qquad \ldots(4)$$

and eqns. (2) and (3) become

$$(\lambda + 2\mu + a_0^2 \rho) \frac{\partial^2 u_1}{\partial x_1^2} - \gamma \left(\frac{\partial \theta}{\partial x_1} + \alpha \frac{\partial^2 \theta}{\partial x_1 \partial t} \right) = \rho \ddot{u}_1 \qquad \ldots(5.1)$$

$$\rho C_v \left(\frac{\partial \theta}{\partial t} + \alpha^* \frac{\partial^2 \theta}{\partial t^2} \right) + \gamma T_0 \frac{\partial^2 u_1}{\partial x_1 \partial t} = K \frac{\partial^2 \theta}{\partial x_1^2} \qquad \ldots(5.2)$$

where $a_0 = \sqrt{(\mu_0 H_3^2 / 4\pi\rho)}$ is the Alfven wave velocity.

For convenience, we shall use notations $u_1 = u$, $x_1 = x$. In vacuum, the system of equations of electrodynamics is expressed as

$$\left(\frac{\partial^2}{\partial x'^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) h_3^0 = 0, \left(\frac{\partial^2}{\partial x'^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) E_2^0 = 0 \quad \ldots(5.3)$$

where $x' = -x$.

The components of Maxwell's stress tensors in elastic medium T_{11}, in vacuum T_{11}^0

$$T_{11} = T_{ij} \mid_{i=j=1} = \frac{\mu_0}{4\pi} [h_i H_j + h_j H_i - \delta_{ij} (\vec{h}.\vec{H})]_{i=j=1}$$

$$= -\frac{\mu_0}{4\pi} h_3 H_3 \quad \ldots(6.1)$$

$$T_{11}^0 = T_{ij}^0 \mid_{i=j=1} = \frac{1}{4\pi} [h_i^0 H_j + h_j^0 H_i - \delta_{ij} (\vec{h}.\vec{H})]_{i=j=1}$$

$$= -\frac{h_3^0 H_3}{4\pi} \quad \ldots(6.2)$$

$$\sigma_{11} = (\lambda + 2\mu) \frac{\partial u}{\partial x} - \gamma (\theta + \alpha \dot{\theta}). \quad \ldots(7)$$

Normal load at the Boundary

The boundary conditions in the case of normal load acting at the plane boundary are given by

$$\sigma_{11} + T_{11} - T_{11}^0 = \sigma_0 H(t), \text{ at } x = x' = 0 \quad \ldots(8)$$

$$E_2 = E_2^0, \text{ at } x = x' = 0 \quad \ldots(9)$$

$$\theta (0, t) = 0, \text{ at } x = x' = 0 \quad \ldots(10)$$

where $H(t)$ is Heaviside function.

Introducing the dimensionless quantities

$$\eta = c_0 x \mid k, \tau = c_0^2 t/k, U = (\lambda + 2\mu + a_0^2 \rho) u/\gamma T_0 k,$$

$$Z = \theta/T_0, \epsilon = \gamma^2 T_0/C_\epsilon (\lambda + 2\mu + a_0^2 \rho), \alpha' = \alpha\omega^*, \alpha^{*\prime} = \alpha^* \omega^*$$

where

$$c_1^2 = (\lambda + 2\mu)/\rho, \; c_0^2 = a_0^2 + c_1^2, \; k = K/\rho C_v, \; \omega^* = \rho C_v c_0^2/K$$

$$C_\epsilon = \rho C_v.$$

Equations (5.1), (5.2) and (5.3) become

$$\frac{\partial^2 U}{\partial \eta^2} - \frac{\partial Z}{\partial \eta} - \alpha' \frac{\partial^2 Z}{\partial \eta \, \partial \tau} - \frac{\partial^2 U}{\partial \tau^2} = 0 \text{ for } \eta > 0 \quad \ldots(11)$$

$$\frac{\partial^2 Z}{\partial \eta^2} - \frac{\partial Z}{\partial \tau} - \alpha^{*\prime} \frac{\partial^2 Z}{\partial \tau^2} - \epsilon \frac{\partial^2 u}{\partial \eta \, \partial \tau} = 0 \text{ for } \eta > 0 \quad \ldots(12)$$

$$\left(\frac{\partial^2 h_3^0}{\partial \eta'^2} - \beta^2 \frac{\partial^2 h_3^0}{\partial \tau^2} \right) = 0 \text{ for } \eta' > 0 \quad \ldots(13)$$

where

$$\eta' = -\eta, \boldsymbol{\beta} = c_0/c.$$

The initial conditions

$$u\,(x, 0) = 0, \theta\,(x, 0) = 0, \frac{\partial u}{\partial x}\,(x, 0) = 0,$$

in the new variables become

$$U\,(\eta, 0) = 0, Z\,(\eta, 0) = 0, \frac{\partial u}{\partial \eta}(\eta, 0) = 0. \quad \ldots(14)$$

The boundary conditions (8), (9) and (10) become

$$\frac{\partial U}{\partial \eta} - Z - \alpha' \frac{\partial Z}{\partial \tau} + \beta_1 h_3^0 H(\tau)/\gamma T_0 = 0 \text{ at } \eta = \eta' = 0 \quad \ldots(15)$$

$$\beta_2 \frac{\partial^2 U}{\partial \tau^2} - \frac{\partial h_3^0}{\partial \eta'} = 0, \text{ at } \eta = \eta' = 0 \quad \ldots(16)$$

where

$$\left.\begin{aligned} h_3 &= -H_3 \frac{\partial u}{\partial x} \\ \beta_2 &= \mu_0 H_3 \gamma T_0/\rho c_0^2 \\ \beta_1 &= H_3/4\pi\gamma T_0 \\ Z\,(0, \tau) &= 0. \end{aligned}\right\} \quad \ldots(17)$$

and

We consider the potential function ϕ defined by

$$U = \frac{\partial \phi}{\partial \eta}. \quad \ldots(18)$$

Putting (18) into (11) and (12), we get

$$Z\,(\eta, \tau) + \alpha' \frac{\partial Z}{\alpha \tau} = \left(\frac{\partial^2}{\partial \eta^2} - \frac{\partial^2}{\partial \tau^2} \right) \phi \text{ for } \eta > 0 \quad \ldots(19)$$

and

$$\frac{\partial^2 Z}{\partial \eta^2} - \frac{\partial Z}{\partial \tau} - \alpha^{*'} \frac{\partial^2 Z}{\partial \tau^2} - \epsilon \frac{\partial^3 \phi}{\partial \tau\, \partial \eta^2} = 0 \text{ for } \eta > 0. \quad \ldots(20)$$

Applying Laplace transform to eqns. (19), (20) and (13) defined by

$$\bar{f}\,(s) = \int_0^\infty f(t)\, e^{-st}\, dt \quad \ldots(21)$$

we obtain

$$(1 + \alpha' s)\,\bar{Z} = \left(\frac{\partial^2}{\partial \eta^2} - s^2\right)\bar{\phi} \text{ for } \eta > 0 \qquad \ldots(22)$$

$$\left(\frac{\partial^2}{\partial \eta^2} - s - \alpha^{*\prime} s^2\right)\bar{Z} = \epsilon\, s\,\frac{\partial^2 \bar{\phi}}{\partial \eta^2} \quad \text{for } \eta > 0 \qquad \ldots(23)$$

$$\bar{h}_3^0 = C_3\, e^{-\beta_5 \eta'} \text{ for } \eta' > 0. \qquad \ldots(24)$$

Using (18) into (14), (15) and (16), we obtain

$$\phi(\eta, 0) = 0,\ \frac{\partial \phi}{\partial \eta}(\eta, 0) = 0,\ \frac{\partial^2 \phi}{\partial \eta^2}(\eta, 0) = 0 \qquad \ldots(25)$$

$$\frac{\partial^2 \phi}{\partial \eta^2} - \left(Z + \alpha' \frac{\partial Z}{\partial \tau}\right) + \beta_1 h_3^0 - \sigma_0 H(\tau)/\gamma T_0 = 0 \text{ for } \eta = 0 \qquad \ldots(26)$$

$$\beta_2 \frac{\partial^3 \phi}{\partial \tau^2 \partial \eta} - \frac{\partial h_3^0}{\partial \eta'} = 0 \text{ for } \eta = \eta' = 0. \qquad \ldots(27)$$

Applying Laplace transform to eqns. (26), (27) and (17), we get

$$\frac{\partial^2 \bar{\phi}}{\partial \eta^2} - (1 + \alpha' s)\,\bar{Z} + \beta_1 \bar{h}_3^0 - \sigma_0/\gamma T_0\, s = 0 \text{ at } \eta = 0 \qquad \ldots(28)$$

$$\beta_2 s^2 \frac{\partial \bar{\phi}}{\partial \eta} - \frac{\partial \bar{h}_3^0}{\partial \eta'} = 0 \text{ at } \eta = \eta' = 0 \qquad \ldots(29)$$

$$\bar{Z}(0, s) = 0 \text{ at } \eta = 0. \qquad \ldots(30)$$

Eliminating $\bar{Z}$ from eqns. (22) and (23), we get

$$\frac{\partial^4 \bar{\phi}}{\partial \eta^4} - [1+\epsilon+s(1 + \alpha^{*\prime} + \epsilon\alpha)]\, s\,\frac{\partial^2 \bar{\phi}}{\partial \eta^2} + s^3(1 + s\alpha^{*\prime})$$
$$\bar{\phi} = 0 \text{ for } \eta > 0. \qquad \ldots(31)$$

The general solution of (31) which vanishes at $\eta \to \infty$ is given by

$$\bar{\phi}(\eta, s) = C_1 e^{-\lambda_1 \eta} + C_2 e^{-\lambda_2 \eta} \text{ for } \eta > 0 \qquad \ldots(32)$$

where λ_1, λ_2 are the roots of eqn. (33)

$$\lambda^4 - s[1 + \epsilon + s(1 + \alpha^{*\prime} + \epsilon\,\alpha')]\,\lambda^2 + s^3(1 + \alpha^{*\prime} s) = 0. \qquad \ldots(33)$$

From equations (22) and (32), we get

$$\bar{Z}(\eta, s) = \frac{1}{(1+\alpha' s)}[C_1(\lambda_1^2 - s^2)\, e^{-\lambda_1 \eta} + C_2(\lambda_2^2 - s^2)\, e^{-\lambda_2 \eta}]$$
$$\text{for } \eta > 0 \qquad \ldots(34)$$

Using (32) into (28), (29) and (30), we get

$$s^2 (C_1 + C_2) + \beta_1 C_3 = \sigma_0/\gamma T_0 s \text{ at } \eta = \eta' = 0 \qquad \text{...(35)}$$

$$\beta_2 s (C_1 \lambda_1 + C_2 \lambda_2) - \beta C_3 = 0 \text{ at } \eta = \eta' = 0 \qquad \text{...(36)}$$

$$C_1 (\lambda_1^2 - s^2) + C_2 (\lambda_2^2 - s^2) = 0 \text{ at } \eta = \eta' = 0. \qquad \text{...(37)}$$

From equations (35), (36) and (37), we obtain

$$C_1 = - \sigma_0 \beta (\lambda_2^2 - s^2)/\gamma T_0 s^2 A, \; C_2 = \sigma_0 \beta (\lambda_1^2 - s^2)/\gamma T_0 s^2 A,$$

$$C_3 = \sigma_0 \beta_2 (\lambda_1 - \lambda_2)(\lambda_1 \lambda_2 + s^2)/\gamma T_0 sA \qquad \text{...(38)}$$

where

$$A = (\lambda_1 - \lambda_2)[\beta_1 \beta_2 s^2 + \beta s (\lambda_1 + \lambda_2) + \beta_1 \beta_2 \lambda_1 \lambda_2]. \qquad \text{...(39)}$$

The equations (38), (32), (34) and (24) provide us

$$\bar{\phi} (\eta, s) = \frac{\sigma_0 \beta [(\lambda_1^2 - s^2) e^{-\lambda_2 \eta} - (\lambda_2^2 - s^2) e^{-\lambda_1 \eta}]}{\gamma T_0 (\lambda_1 - \lambda_2)[\beta_1 \beta_2 s^2 + \beta s (\lambda_1 + (\lambda_2) + \beta_1 \beta_2 \lambda_1 \lambda_2]}$$

$$\eta > 0 \qquad \text{...(40)}$$

$$\bar{Z} (\eta, s) = \frac{\sigma_0 \beta (\lambda_1^2 - s^2)(\lambda_2^2 - s^2)(e^{-\lambda_1 \eta} - e^{-\lambda_2 \eta})}{\gamma T_0 s^2 (1 + \alpha' s)(\lambda_1 - \lambda_2)[\beta_1 \beta_2 s^2 + \beta s (\lambda_1 + \lambda_2) + \beta_1 \beta_2 \lambda_1 \lambda_2]}$$

$$\eta > 0 \qquad \text{...(41)}$$

$$\bar{h}_3^0 (\eta', s) = \frac{\sigma_0 \beta_2 (\lambda_1 \lambda_2 + s^2) e^{-\beta \eta' s}}{\gamma T_0 s [\beta_1 \beta_2 s^2 + \beta s (\lambda_1 + \lambda_2) + \beta_1 \beta_2 \lambda_1 \lambda_2]} \; \eta' > 0. \qquad \text{...(42)}$$

The transformed displacement is given by

$$\bar{U} (\eta, s) = \frac{\sigma_0 \beta [\lambda_1 (\lambda_2^2 - s^2) e^{-\lambda_1 \eta} - \lambda_2 (\lambda_1^2 - s^2) e^{-\lambda_2 \eta}]}{\gamma T_0 s^2 (\lambda_1 - \lambda_2)[\beta_1 \beta_2 s^2 + \beta s (\lambda_1 + \lambda_2) + \beta_1 \beta_2 \lambda_1 \lambda_2]}$$

$$\eta > 0. \qquad \text{...(43)}$$

3. Small Time Approximations

The dependence of λ_1, λ_2 on s is very complicated and hence the inversion of the Laplace transform is difficult. These difficulties, however, reduce if we use some approximate methods. As the thermal relaxation effects are short-lived so we confine our discussions to small time approximations, i. e., we take s large. A similar approach was used by Sharma[6] to study the thermal shock problem in generalised theory[7] of thermoelasticity. Now the roots λ_1 and λ_2 of eqn. (33) are given by

$$\lambda_{1,2} = s V_{1,2}^{-1} + B_{1,2} + D_{1,2} (1/s) + O(1/s^2) \qquad \text{...(44)}$$

where

$$V_{1,2}^{-1} = (K_2 \pm \Gamma^{1/2})^{1/2}/\sqrt{2} \qquad \text{...(45a)}$$

$$B_{1,2} = [K_1 \pm (K_1 K_2 - 2)/\sqrt{2}]/2\sqrt{2}\,(K_2 \pm \Gamma^{1/2})^{1/2} \qquad \ldots(45b)$$

$$D_{1,2} = \{\pm K_1^2/\Gamma^{1/2} \mp (K_1 K_2 - 2)^2/\Gamma^{3/2} - (K_1 \pm (K_1 K_2 - 2)/\Gamma^{1/2})^2/$$
$$2\,(K_2 \pm \Gamma^{1/2})\}/4\sqrt{2}\,(K_2 \pm \Gamma^{1/2})^{1/2} \qquad \ldots(45c)$$

and

$$\Gamma = K_2^2 - 4\alpha^{*\prime} = (1 + \epsilon\alpha' - \alpha^{*\prime})^2 + 4\,\epsilon\,\alpha'\,\alpha^{*\prime} \qquad \ldots(45d)$$

$$K_1 = 1 + \epsilon,\; K_2 = 1 + \epsilon\,\alpha' + \alpha^{*\prime}. \qquad \ldots(45e)$$

Again $(1 + \epsilon\alpha' + \alpha^{*\prime})^2 > \Gamma$ so that $1/V_1^2 > 1/V_2^2$ or $V_1 < V_2$.

Thus V_1 corresponds to the speed of the slowest wave and V_2 to that of fastest wave. Therefore, the points of the solid for which $\eta > V_2\,\tau$ do not experience any disturbance. Also from equations (45) we see that as $\alpha' = \alpha^{*\prime} = 0$, $V_1 \to 1$ and $V_2 \to \infty$. But this corresponds to the case of conventional coupled theory of thermoelasticity, which predicts an infinite speed of heat propagation. Thus we conclude that the wave propagating with speed V_1 must be elastic and that propagating with speed V_2 is the thermal wave. The third wave travelling with velocity c_0 as the Alfven acoustic wave.
Equations (40), (41), (42), and (43) with the help of (44) provide us

$$\bar{\phi}\,(\eta, s) = \frac{\sigma_0\beta}{\gamma T_0}\Bigg[\Bigg\{\frac{(1 - V_1^2)}{V_1^2}\,\frac{P'}{s^3} + \Bigg(\frac{2B_1\,P'}{V_1} + \frac{Q'\,(1 - V_1^2)}{V_1^2}\Bigg)\frac{1}{s^4}$$
$$+ O\Bigg(\frac{1}{s^5}\Bigg)\Bigg\}\,e^{-\lambda_2\eta} - \Bigg\{\frac{(1 - V_2^2)\,P'}{V_2^2\,s^3} + \Bigg(\frac{2B_2\,P'}{V_2}$$
$$+ \frac{Q'\,(1 - V_2^2)}{V_2^2}\Bigg)\frac{1}{s^4} + O\Bigg(\frac{1}{s^5}\Bigg)\Bigg\}\,e^{-\lambda_1\eta}\Bigg] \qquad \ldots(46)$$

$$\bar{Z}\,(\eta, s) = \frac{\sigma_0\,\beta}{\gamma T_0\alpha'}\Bigg[\frac{M_1\,P'}{s^2} + (M_1\,Q' + M_2\,P' - \frac{M_1\,P'}{\alpha'})\,\frac{1}{s^3}$$
$$+ O\Bigg(\frac{1}{s^4}\Bigg)\Bigg][e^{-\lambda_2\eta} - e^{-\lambda_1\eta}] \qquad \ldots(47)$$

$$\bar{h}_3^0\,(\eta\; s) = \frac{\sigma_0\,\beta_2}{\gamma\,T_0}\Bigg[\Bigg(\frac{1 + V_1\,V_2}{V_1\,V_2}\Bigg)\frac{P}{s} + \Bigg\{\Bigg(\frac{1 + V_1\,V_2}{V_1\,V_2}\Bigg)Q$$
$$+ \frac{(B_1\,V_1 + B_2\,V_2)P}{V_1\,V_2}\Bigg\}\frac{1}{s^2} + O\Bigg(\frac{1}{s^3}\Bigg)\Bigg]\,e^{-B_3\eta'} \qquad \ldots(48)$$

$$\bar{U}\,(\eta, s) = \frac{\sigma_0\,\beta}{\gamma T_0}\Bigg[\Bigg\{\Bigg(\frac{1 - V_2^2}{V_1\,V_2^2}\Bigg)\frac{P'}{s^2} + \Bigg\{\Bigg(\frac{B_1\,(1 - V_2^2)}{V_2^2} + \frac{2B_2}{V_1 V_2}\Bigg)P'$$
$$+ \Bigg(\frac{1 - V_2^2}{V_1\,V_2^2}\Bigg)Q'\Bigg\}\frac{1}{s^3} + O\Bigg(\frac{1}{s^4}\Bigg)\Bigg\}\,e^{-\lambda_1\eta}$$
$$- \Bigg\{\Bigg(\frac{1 - V_1^2}{V_1^2\,V_2}\Bigg)\frac{P'}{s^2} + \Bigg\{\Bigg(\frac{B_2\,(1 - V_1^2)}{V_1^2} + \frac{2B_1}{V_1\,V_2}\Bigg)P'$$
$$+ \Bigg(\frac{1 - V_1^2}{V_1^2\,V_2}\Bigg)Q'\Bigg\}\frac{1}{s^3} + O\Bigg(\frac{1}{s^4}\Bigg)\Bigg\}\,e^{-\lambda_2\eta}\Bigg] \qquad \ldots(49)$$

where

$$P = \frac{1}{\beta_1 \beta_2}\Bigg[1 + \frac{\beta^2 (V_1 + V_2)^2}{\beta_1^2 \beta_2^2 V_1^2 V_2^2} + \frac{2\beta (V_1 + V_2)}{\beta_1 \beta_2 V_1^2 V_2^2} - \frac{\beta^3 (V_1 + V_2)^3}{\beta_1^3 \beta_2^3 V_1^3 V_2^3}$$

$$- \frac{1}{V_1 V_2} + \frac{1}{V_1^2 V_2^2} + \frac{\beta^4 (V_1 + V_2)^4}{\beta_1^4 \beta_2^4 V_1^4 V_2^4}$$

$$- \frac{3\beta^2 (V_1 + V_2)^2}{\beta_1^2 \beta_2^2 V_1^3 V_2^3} - \frac{\beta (V_1 + V_2)}{\beta_1 \beta_2 V_1 V_2}\Bigg] \qquad \ldots(50)$$

$$Q = \frac{1}{\beta_1 \beta_2}\Bigg[\frac{2\beta (B_1 + B_2)(V_1 + V_2)}{\beta_1^2 \beta_2^2 V_1 V_2} - \frac{(B_1 V_1 + B_2 V_2)}{V_1 V_2}$$

$$+ \frac{2\beta}{\beta_1 \beta_2}\left\{\frac{(V_1 + V_2)(B_1 V_1 + B_2 V_2)}{V_1^2 V_2^2} + \frac{(B_1 + B_2)}{V_1 V_2}\right\}$$

$$- \frac{3\beta^3 (V_1 + V_2)^2 (B_1 + B_2)}{\beta_1^3 \beta_2^3 V_1^2 V_2^2} + \frac{2 (B_1 V_1 + B_2 V_2)}{V_1 V_2} - \frac{3\beta}{\beta_1^2 \beta_1^2}$$

$$\times \left\{\frac{(V_1 + V_2)^2 (B_1 V_1 + B_2 V_2)}{V_1^3 V_2^3} + \frac{2 (B_1 + B_2)(V_1 + V_2)}{V_1^2 V_2^2}\right\}$$

$$- \frac{4\beta^4 (V_1 + V_2)^3 (B_1 + B_2)}{\beta_1^4 \beta_2^4 V_1^3 V_2^3} - \frac{\beta (B_1 + B_2)}{\beta_1 \beta_2}\Bigg] \qquad \ldots(51)$$

$$\left.\begin{aligned} M_1 &= \frac{1 + V_1 V_2}{V_1 V_2} - \frac{(V_1^2 + V_2^2)}{V_1^2 V_2^2} \\ M_2 &= \frac{2B_2}{V_1^2 V_2^2} - \frac{2 (B_1 V_2 + B_2 V_1)}{V_1 V_2} \end{aligned}\right\} \qquad \ldots(52)$$

and

$$P' = V_1 V_2 P/(V_2 - V_1),\ Q' = [V_1 V_2 Q/V_2 - V_1)] - [V_1^2 V_2^2 (B_1 - B_2) \\ P/(V_2 - V_1)^2]. \qquad \ldots(53)$$

Inverting the Laplace transforms for small times, i.e., for large s, eqns. (46), (47), (48) and (49) provide us

$$\phi (\eta, \tau) = \frac{\sigma_0 \beta}{\gamma T_0}\Bigg[\Bigg\{\left(\frac{1 - V_1^2}{V_1^2}\right) P' (\tau - \eta/V_2)^2 H (\tau - \eta/V_2)$$

$$+ \left(\frac{2B_1 P'}{V_1} + \frac{1 - V_1^2}{V_1^2} Q'\right)(\tau - \eta/V_2)^3 H (\tau - \eta/V_2)\Bigg\} e^{-\beta_2 \eta}$$

$$- \Bigg\{\left(\frac{1 - V_2^2}{V_2^2}\right) P' (\tau - \eta/V_1)^2 H (\tau - \eta/V_1)$$

$$+ \left(\frac{2B_2 P'}{V_2} + \frac{1 - V_2^2}{V_2^2} Q'\right)(\tau - \eta/V_1)^3 H (\tau - \eta/V_1)\Bigg\}$$

$$e^{-\beta_1 \eta}\Bigg] \qquad \ldots(54)$$

$$Z(\eta, \tau) = \frac{\sigma_0 \beta}{\gamma T_0 \alpha'} [\{P' M_1 (\tau - \eta/V_2) H(\tau - \eta/V_2) + (M_1 Q' + M_2 P'$$

$$- \frac{M_1 P'}{\alpha'}) (\tau - \eta/V_2)^2 H(\tau - \eta/V_2)\} e^{-B_2 \eta} - \{P' M_1$$

$$(\tau - \eta/V_1) H(\tau - \eta/V_1) + (M_1 Q' + M_2 P' - \frac{M_1 P'}{\alpha'})$$

$$(\tau - \eta/V_1)^2 H(\tau - \eta/V_1)\} e^{-B_1 \eta}] \qquad \ldots(55)$$

$$h_3^0(\eta', \tau) = \frac{\sigma_0 \beta_2}{\gamma T_0} \Big[\Big(\frac{1 + V_1 V_2}{V_1 V_2} \Big) P\, H(\tau - \beta\eta') + \Big\{ \Big(\frac{1 + V_1 V_2}{V_1 V_2} \Big)$$

$$\times Q + \Big(\frac{B_1 V_1 + B_2 V_2)P}{V_1 V_2} \Big\} (\tau - \beta\eta')\, H(\tau - \beta\eta') \Big] \qquad \ldots(56)$$

$$U(\eta, \tau) = \frac{\sigma_0 \beta}{\gamma T_0} \Big[\Big\{ \Big(\frac{1 - V_2^2}{V_1 V_2^2} \Big) P' (\tau - \eta/V_1) H(\tau - \eta/V_1)$$

$$+ \Big\{ \Big(\frac{B_1 (1 - V_2^2)}{V_2^2} + \frac{2B_2}{V_1 V_2} \Big) P' + \frac{1 - V_2^2}{V_2^2} Q' \Big\}$$

$$\times (\tau - \eta/V_1)^2 H(\tau - \eta/V_1) \Big\} e^{-B_1 \eta}$$

$$- \Big\{ \Big(\frac{1 - V_1^2}{V_1^2 V_2} \Big) P' (\tau - \eta/V_2) H(\tau - \eta/V_2) + \Big\{ \frac{B_1 (1 - V_1^2)}{V_1^2}$$

$$+ \frac{2B_2}{V_1 V_2} \Big) P' + \frac{1 - V_1^2}{V_1^2} Q' \Big\} (\tau - \eta/V_2) H(\tau - \eta/V_2) \Big\} e^{-B_2 \eta} \Big] \qquad \ldots(57)$$

4. Thermal Shock at the Boundary

The thermal shock $\theta(0, t) = \theta_0 H(t)$ also produces disturbances in the elastic medium in the presence of magnetic field. In this case the boundary conditions are given by

$$\left.\begin{aligned} &\sigma_{11} + T_{11} - T_{11}^0 = 0 \text{ at } x = x' = 0 \\ &E_2 = E_2^0 \text{ at } x = x' = 0 \\ &\theta(0, t) = \theta_0 H(t) \text{ at } x = x' = 0 \end{aligned}\right\} \qquad \ldots(58)$$

The transformed potential function $\bar{\phi}$, temperature $\bar{Z}$, induced magentic field $\bar{h}_3^0$, and displacement $\bar{U}$, are obtianed as

$$\bar{\phi}(\eta, s) = \frac{\theta_0 (1 + \alpha' s) [(s\beta + \beta_1 \beta_2 \lambda_2) e^{-\lambda_1 \eta} - (s\beta + \beta_1 \beta_2 \lambda_1) e^{-\lambda_2 \eta}]}{s T_0 (\lambda_1 - \lambda_2) [\beta_1 \beta_2 s^2 + \beta s (\lambda_1 + \lambda_2) + \beta_1 \beta_2 \lambda_1 \lambda_2]}$$

$$\text{for } \eta > 0 \qquad \ldots(59)$$

$$\bar{Z}(\eta, s) = \frac{\theta_0 [(\lambda_1^2 - s^2)(\beta s + \beta_1 \beta_2 \lambda_2) e^{-\lambda_1 \eta} - (\lambda_2^2 - s^2)(\beta s + \beta_1 \beta_2 \lambda_1) e^{-\lambda_2 \eta}]}{s T_0 (\lambda_1 - \lambda_2)[\beta_1 \beta_2 s^2 + \beta s(\lambda_1 + \lambda_2) + \beta_1 \beta_2 \lambda_1 \lambda_2]}$$

$$\text{for } \eta > 0 \quad \ldots(60)$$

$$\bar{h}_3^0 (\eta', s) = \frac{\theta_0 (1 + \alpha' s) \beta_2 s \, e^{-B_3 \eta'}}{T_0 [\beta_1 \beta_2 s^2 + \beta s (\lambda_1 + \lambda_2) + \beta_1 \beta_2 \lambda_1 \lambda_2]} \text{ for } \eta' > 0 \quad \ldots(61)$$

$$\bar{U}(\eta, s) = \frac{\theta_0 (1 + \alpha' s)[\lambda_2 (\beta s + \beta_1 \beta_2 \lambda_1) e^{-\lambda_2 \eta} - \lambda_1 (\beta s + \beta_1 \beta_2 \lambda_2) e^{-\lambda_1 \eta}]}{T_0 s (\lambda_1 - \lambda_2)[\beta_1 \beta_2 s^2 + \beta s (\lambda_1 + \lambda_2) + \beta_1 \beta_2 \lambda_1 \lambda_2]}$$

$$\text{for } \eta > 0. \quad \ldots(62)$$

Inverting the Laplace transform for small times, eqns. (59), (60), (61), and (62) provide us

$$\phi(\eta, \tau) = \frac{\theta_0}{T_0} \Big[\Big\{ P^* (\tau - \eta/V_1) H(\tau - \eta/V_1) + Q^* (\tau - \eta/V_1)^2$$
$$\times H(\tau - \eta/V_1) \} e^{-B_1 \eta} - \{ P^{*\prime} (\tau - \eta/V_2) H(\tau - \eta/V_2)$$
$$+ Q^{*\prime} (\tau - \eta/V_2)^2 H(\tau - \eta/V_2) \} e^{-B_2 \eta} \Big] \quad \ldots(63)$$

$$Z(\eta, \tau) = \frac{\theta_0}{T_0} [\{ P' N_1 H(\tau - \eta/V_1) + (N_1 Q' + N_2 P')(\tau - \eta/V_1)$$
$$H(\tau - \eta/V_1) \} e^{-B_1 \eta} - \{ P' N_1' H(\tau - \eta/V_2) + (N_1'' Q'$$
$$+ N_2' P')(\tau - \eta/V_2) H(\tau - \eta/V_2) \} e^{-B_2 \eta}] \quad \ldots(64)$$

$$h_3^0 (\eta', \tau) = \frac{\theta_0 \beta_2}{T_0} [\alpha' P \delta(\tau - \beta \eta') + (P + Q\alpha') H(\tau - \beta \eta')] \quad \ldots(65)$$

$$U(\eta, \tau) = \frac{\theta_0}{T_0} [\{ M_1' H(\tau - \eta/V_2) + M_2' (\tau - \eta/V_2) H(\tau - \eta/V_2) \} e^{-B_2 \eta}$$
$$- \{ M_1'' H(\tau - \eta/V_1) + M_2'' (\tau - \eta/V_1) H(\tau - \eta/V_1) \} e^{-B_1 \eta}]$$
$$\ldots(66)$$

where

$$P^* = P' (\beta_1 \beta_2 \alpha' + \alpha' \beta V_2)/V_2, \; Q^* = [P' (\beta V_2 + \beta_1 \beta_2 B_2 V_2 + \beta_1 \beta_2)/V_2]$$
$$+ Q' (\beta_1 \beta_2 \alpha' + \alpha' \beta V_2)/V_2,$$

and

$$P^{*\prime} = P' (\beta_1 \beta_2 \alpha' + \alpha' \beta V_1)/V_1, \; Q^{*\prime} = [P' (\beta V_1 + \beta_1 \beta_2 B_1 V_1$$
$$+ \beta_1 \beta_2)/V_1] + Q' (\beta_1 \beta_2 \alpha' + \alpha' \beta V_1)/V_1$$

$$N_1 = [\beta V_2 + \beta_1 \beta_2 (1 - V_1^2)]/V_1^2 V_2, \; N_2 = [2 B_1 (\beta V_2 + \beta_1 \beta_2)/V_1 V_2]$$
$$+ (1 - V_1^2) \beta_1 \beta_2 B_2 / V_1^2 V_2$$

$$N_1' = (\beta V_1 + \beta_1 \beta_2 (1 - V_2^2))/V_1 V_2^2, \; N_2' = [2B_2(\beta V_1 + \beta_1\beta_2)/V_1 V_2]$$

$$+ (1 - V_2^2)\, \beta_1\, \beta_2\, B_1/V_1 V_2^2$$

$$M_1' = \beta \alpha' P'/V_2, \; M_2' = [\beta (P' + \alpha' Q')/V_1] + \alpha' P' (\beta_1 \beta_2 B_1 V_1$$

$$+ \beta_1 \beta_2 V_1 V_2 \beta_1 \beta_2 V_2)/V_1 V_2$$

$$M_1'' = \beta \alpha' P'/V_1, \; M_2'' = [\beta (P' + \alpha' Q')/V_1] + \alpha' P'(\beta_1 \beta_2 B_2 V_2$$

$$+ \beta_1 \beta_2 V_1 V_2 + \beta_1 \beta_2 B_1 V_1)/V_1 V_2.$$

The stresses in the vacuum and the elastic medium can be easily obtained by using various expressions in eqns. (6) and (7).

5. Discussion of the Results

The short time solutions above show that they consist of three waves, i. e., the elastic wave, thermal wave, and Alfven-acoustic wave travelling with velocities V_1, V_2, and c_0 respectively. The terms containing $H(\tau - \eta/V_1)$ represent the contribution of the elastic wave in vicinity of its wavefront ($\eta = V_1 \tau$), the terms with $H(\tau - \eta/V_2)$ represent the contribution of the thermal wave in the vicinity of its wavefront ($\eta = V_2 \tau$), and those with $H(\tau - \beta\eta')$ represent the contribution of the Alfven acoustic wave in the vicinity of its wavefront ($\eta' = \tau/\beta$). The displacement and temperature are found to be continuous at the wave fronts and the perturbed magnetic is discontinuous in case of normal load. The discontinuty is given by

$$\left[h_3^{0+} - h_3^{0-}\right]_{\eta' = \tau/\beta} = \sigma_0 \beta_2 (1 + V_1 V_2) P/V_1 V_2 \gamma T_0.$$

In case of thermal shock the deformation, temperature, and the perturbed magnetic field are all found to be discontinuous and the jumps at the wave-fronts are given by

$$[U^+ - U^-]_{\eta = V_1 \tau} = -\theta_0 M_1'' \exp(-B_1 V_1 \tau)/T_0,$$

$$[U^+ - U^-]_{\eta = V_2 \tau} = \theta_0 M_1' \exp(-B_2 V_2 \tau)/T_0,$$

$$[Z^+ - Z^-]_{\eta = V_1 \tau} = \theta_0 N_1 P' \exp(-B_1 V_1 \tau)/T_0$$

$$[Z^+ - Z^-]_{\eta = V_2 \tau} = -\theta_0 N_1' P' \exp(-B_2 V_2 \tau)/T_0$$

$$\left[h_3^{0+} - h_3^{0-}\right]_{\eta' = \tau/\beta} = \theta_0 \beta_2 (P + Q \alpha')/T_0.$$

Clearly the discontinuities in deformation and temperature decay exponentially with time. In case of conventional coupled theory of thermoelasticity $\alpha = \alpha^* = 0$ and hence

$$K_1 = 1 + \epsilon,\ K_2 = 1,\ V_1 = 1,\ V_2 \to \infty,\ \Gamma = 1,\ B_1 = \epsilon/2$$

$$B_2 \to \infty,\ D_1 = \epsilon\,(4 - \epsilon)/8,\ D_2 \to \infty.$$

It is observed that the perturbed magnetic field experiences finite and infinite jumps in case of normal load and thermal shock, respectively. In case of normal load the displacement and temperature are found to be continuous at both the wave fronts whereas in case of thermal shock these quantities are continuous at the thermal wave front but experience finite jumps at the elastic wave fronts.

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THERMO-CREEP TRANSITION OF A THICK ISOTROPIC SPHERICAL SHELL UNDER INTERNAL PRESSURE

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Creep stresses for a thick isotropic spherical shell under internal pressure and steady state of temperature have been derived. Results are depicted graphically. It is seen that shells made of incompressible material require higher pressure to yield as compared to shells made of compressible material. For no thermal effects, the results are the same as given by Hulsurkar[1] and Bailey[2].

1. INTRODUCTION

The problem of elastic-plastic and creep of spherical shells under internal pressure have been discussed by Bailey[2] and the effects of steady state of temperature on the above problem has been discussed by Derrington and Johnson[3]. These authors have analysed the problem after making some simplifying assumptions, like infinitesimal deformation and incompressibility of the material. Additionally, these works are based on the use of a yield condition and creep strain laws. Seth[9,10] transition theory does not require these assumptions. It introduces the concept of generalized strain measures and then finds a solution of governing differential equations near the transition points. It has been shown by Hulsurkar[1], Seth[13,14], Gupta and Dharmani[4–6] that the asymptotic solution through the principal stress-difference gives the creep stresses. Seth[9,10] has defined the generalized principal strain measure as,

$$e_{ii} = \int_0^{e_{ii}^A} \left[1 - 2e_{ii}^A\right]^{(n-2)/2} d\, e_{ii}^A = \frac{1}{n}\left[1 - (1 - 2e_{ii}^A)^{n/2}\right] \qquad \ldots(1.1)$$

where n is the measure and e_{ii}^A is the principal Almansi strain components. In cartesian framework we can rapidly write down the generalized measure in terms of any other measure. In terms of the principal Almansi strain components e_{ii}^A, the generalized principal strain components e_{ii}^M are,

$$e_{ii}^M = \left[\frac{1}{n}\{1 - (1 - 2e_{ii}^A)^{n/2}\}\right]^m. \qquad \ldots(1.2)$$

For uniaxial case it is given by

$$e = \left[\frac{1}{n} \left\{ 1 - \left(\frac{l_0}{l} \right)^n \right\} \right]^m \qquad \ldots(1.3)$$

where m is the irreversibility index and l_0 and l are the initial and strained lengths respectively.

In this paper, creep stresses for a thick isotropic spherical shell under internal pressure and steady state of temperature have been derived by using the concept of generalized strain measures and asymptotic solution through the principal stress-difference.

2. Governing Equations

Consider a spherical shell of internal and external radii a and b respectively, subjected to uniform internal pressure p and a steady state of temperature θ applied on the internal surface of the shell. Due to spherical symmetry of the structure, the components of displacements in spherical co-ordinates (r, ϕ, z) are given by Seth[8],

$$u = r(1 - \beta), \; v = 0, \; w = 0 \qquad \ldots(2.1)$$

where β is a function of $r = (x^2 + y^2 + z^2)^{1/2}$ only.

The generalized components of strain from equation (1.2) are

$$e_{rr} = \frac{1}{nm} [1 - (r\beta' + \beta)^n]^m$$

$$e_{\phi\phi} = \frac{1}{nm} [1 - \beta^n]^m = e_{zz}$$

$$e_{r\phi} = e_{\phi z} = e_{zr} = 0 \qquad \ldots(2.2)$$

where

$$\beta^1 = \frac{d\beta}{dr}.$$

The thermo-elastic stress-strain relation for isotropic materials are given by Parkus[12] and Fung[11] as

$$\tau_{ij} = \lambda \delta_{ij} I_1 + 2\mu e_{ij} - \xi\theta\delta_{ij} \; (i, j = 1, 2, 3) \qquad \ldots(2.3)$$

where λ and μ are the Lame's constant and $\xi = \alpha(3\lambda + 2\mu)$, α being the coefficient of thermal expansion, while θ denotes the steady state temperature. Further θ has to satisfy

$$\nabla^2\theta = 0. \qquad \ldots(2.4)$$

Using eqn. (2.2) in eqn. (2.3), we get

$$\tau_{rr} = \frac{(\lambda+2\mu)}{nm}[1 - (r\beta' + \beta)^n]^m + \frac{2\lambda}{nm}[1 - \beta^n]^m - \xi\,\theta$$

$$\tau_{\phi\phi} = \tau_{zz} = \frac{\lambda}{nm}[1 - (r\beta' + \beta)^n]^m + \frac{2(\lambda+\mu)}{nm}[1 - \beta^n]^m - \xi\theta$$

$$\tau_{r\phi} = \tau_{\phi z} = \tau_{zr} = 0. \qquad \text{...(2.5)}$$

The equation of equilibrium are all satisfied except

$$\frac{d(\tau_{rr})}{dr} + \frac{2(\tau_{rr} - \tau_{\phi\phi})}{r} = 0. \qquad \text{...(2.6)}$$

The temperture field satisfyin eqn. (2.4) and

$$\theta = \theta_0 \text{ at } r = a$$

$$\theta = 0 \text{ at } r = b \qquad \text{...(2.7)}$$

where θ_0 is a constant, is given by

$$\theta = \frac{\theta_0\, a}{(b-a)}(b/r - 1). \qquad \text{...(2.8)}$$

Using eqns. (2.5) and (2.8) in eqn. (2.6), we have a non-linear differential equation in β, as

$$P(P+1)^{n-1}\beta\frac{dP}{d\beta}[1 - \beta^n(P+1)^n]^{m-1} + P(P+1)^n[1 - \beta^n$$
$$(P+1)^n]^{m-1} + 2(1-c)P(1-\beta^n)^{m-1}$$
$$+ \frac{c\,\xi\,\bar{\theta}_0\, n^m}{2\mu\, r\,\beta^n\, m\, n} - \frac{2c}{m\, n\,\beta^n}[\{1 - \beta^n(P+1)^n\}^m - (1-\beta^n\}^m]$$
$$= 0 \qquad \text{...(2.9)}$$

where

$$r\beta' = p\,\beta,\; c = \frac{2\mu}{2\mu+\lambda} \text{ and } \bar{\theta}_0 = -\frac{\theta_0\, ab}{(b-a)}.$$

For $m = 1$, which holds good for secondary stage of creep[5], eqn. (2.9) reduces to

$$\left[(P + \frac{2c}{n})(P+1)^n + 2P(1-c) - \frac{2c}{n} + \frac{c\xi\,\bar{\theta}_0}{2\mu r\beta^n}\right]\frac{d\beta}{dP}$$
$$+ \beta P(P+1)^{n-1} = 0. \qquad \text{...(2.10)}$$

The transition points of β in eqn. (2.9) are $P \to 0$, $P \to -1$ and $P \to \pm\infty$. The only critical point of interest is $P \to -1$ and $P \to \pm\infty$. The case of transition point $P \to \pm\infty$ is discussed by Gupta and Rana[7] which gives the plastic stresecs.

The boundary conditions are

$$\tau_{rr} = -p \text{ at } r = a$$
$$= 0 \text{ at } r = b. \qquad \ldots(2.11)$$

3. Asymptotic Solution Through $P \to -1$

For creep stresses, we define the transition function R_1 through the principal stress-difference (see Seth[13,14], Hulsurkar[1], Gupta and Dharmani[4-6]) as

$$R_1 = \tau_{rr} - \tau_{\phi\phi} \equiv \frac{2\mu}{nm}\left[\{1 - \beta^n (P+1)^n\}^m - \{1 - \beta^n\}^m \right]. \qquad \ldots(3.1)$$

Taking logarithmic differentiation of equation (3.1), with respect to β, we get

$$\frac{d}{d\beta} \log R_1 = mn\beta^{n-1} \frac{[(1-\beta^n)^{m-1} - \{1-\beta^n(P+1)^n\}^{m-1}\{(P-1)^n + (P+1)^{n-1}\beta \frac{dP}{d\beta}\}]}{[\{1-\beta^n (P+1)^n\}^m - \{1-\beta^n\}^m]}. \qquad \ldots(3.2)$$

Substituting the value of $\frac{dP}{d\beta}$ from eqn. (2.9) in eqn. (3.2), we have

$$\frac{d}{d\beta} \log R_1 = mn\beta^{n-1} \frac{[\{1-\beta^n\}^{m-1} + 2(1-c)\{1-\beta^n\}^{m-1} + \frac{c\xi\bar{\theta}_0 n^m}{2\mu r\beta^n mn} - \frac{2c}{mn\beta^n P}\{(1-\beta^n(P+1)^n)^m - (1-\beta^n)^m\}]}{[\{1 - \beta^n (P+1)^n\}^m - \{1 - \beta^n\}^m]}.$$

The asymptotic value of eqn. (3.3), as $P \to -1$, is

$$\frac{d}{d\beta} \log R_1 = \frac{mn\beta^{n-1}(3-2c)(1-\beta^n)^{m-1}}{\{1-(1-\beta^n)^m\}} - \frac{c\,\xi\,\bar{\theta}_0 n^m}{2\mu\, r\beta\,\{1-(1-\beta^n)^m\}} + \frac{2c}{\beta}. \qquad \ldots(3.4)$$

Integration of eqn. (3.4) gives

$$R_1 = A_0\, r^{-2c}\,[1 - (1-\beta^n)^m]^{3-2c} \exp(f_1) \qquad \ldots(3.5)$$

where A_0 is a constant of integration and

$$f_1 = \frac{c\xi\,\bar{\theta}_0}{2\mu} \int \frac{dr}{r^2\,\{1-(1-\beta^n)^m\}}.$$

The asymptotic value of β as $P \to -1$ is D/r, D being a constant, therefore eqn. (3.5) becomes,

$$R_1 = \tau_{rr} - \tau_{\phi\phi} \equiv A_0\, r^{-2c}\,[1 - (1 - D^n r^{-n})^m]^{3-2c} \exp(f_1). \qquad \ldots(3.6)$$

Using eqn. (3.6) in eqn (2.6), and integrating, we get

$$\tau_{rr} = -2A_0 \int r^{-2c-1}\,[1 - (1 - D^n r^{-n})^m]^{3-2c} \exp(f_1),\, \theta dr + A_1 \qquad \ldots(3.7)$$

where A_1 is a constant of integration.

Using boundary conditions (2.11) in eqn. (3.7), we have

$$A_1 = [2A_0 \int r^{-2c-1} \{1 - (1 - D^n r^{-n})^{m}\}^{3-2c}. \exp(f_1)\, dr]_{r=b}$$

and

$$A_0 = \frac{-p}{2 \int_a^b r^{-2c-1} \{1 - (1 - D^n r^{-n})^m\}^{3-2c}. \exp(f_1)\, dr}.$$

Substituting the value of A_0 and A_1 in equations (3.6) and (3.7), we get

$$\tau_{rr} = -p \frac{\int_r^b r^{-2c-1}[1-(1-D^n r^{-n})^m]^{3-2c}. \exp(f_1)\, dr}{2 \int_a^b r^{-2c-1} [1 - (1 - D^n r^{-n})^m]^{3-2c}. \exp(f_1)\, dr}$$

$$\tau_{\phi\phi} = \tau_{zz} = \tau_{rr} + \frac{pr^{-2c} [1 - (1 - D^n r^{-n})^m]^{3-2c}. \exp(f_1)}{2 \int_a^b r^{-2c-1} [1 - (1 - D^{-n} r^{n})^m]^{3-2c}. \exp(f_1)\, dr} \quad \text{...(3.8)}$$

Equation (3.7) corresponds to only one stage of creep. If all the three stages of creep are to be taken into account, we shall add the incremental values[1,13,14] of $\tau_{rr} - \tau_{\phi\phi}$. Thus from eqn. (3.7), we have

$$\tau_{rr} - \tau_{\phi\phi} = A_0\, r^{-6c} \prod_{m,n} [1 - (1-D^n r^{-n})^m]^{3-2c}. \exp(f_1) \quad \text{...(3.9)}$$

where m, n having three different sets of values each corresponding to one stage of creep and the transitional creep stresses are given by

$$\tau_{rr} = -p \frac{\int_r^b r^{-6c-3} \prod_{m,n} [1 - (1 - D^n r^{-n})^m]^{3-2c}. \exp(f_1)\, dr}{\int_a^b r^{-6c-3} \prod_{m,n} [1 - (1 - D^n r^{n})^m]^{3-2c}. \exp(f_1)\, dr}$$

$$\tau_{\phi\phi} = \tau_{zz} = \tau_{rr} + \frac{p\, r^{-6c-3} \prod_{m,n} [1 - (1 - D^n r^{-n})^m]^{3-2c}. \exp(f_1)\, dr}{\int_a^b r^{-6c-3} \prod_{m,n} [1 - (1 - D^n r^{-n})^m]^{3-2c} \exp(f_1)\, dr}$$

...(3.10)

where

$$f_1 = \frac{c\, \xi \bar{\theta}_0}{2\mu} \int \frac{dr}{\prod_{m,n} r^2 [1 - (1 - D^n) r^{-n})^m]}.$$

4. Shell under Steady State of Creep

Transitional creep stresses for secondary state of creep are obtained by putting $m = 1$ in eqn. (3.8) as

$$\tau_{rr} = -p \frac{\int_r^b r^{-3n+2c(n-1)-1} . \exp(f_1)\, dr}{\int_a^b r^{-3n+2c(n-1)-1} . \exp(f_1)\, dr}$$

$$\tau_{\phi\phi} = \tau_{zz} = \tau_{rr} + \frac{p\, r^{-3n+2c(n-1)-1} . \exp(f_1)}{2 \int_a^b r^{-3n+2c(n-1)-1} . \exp(f_1)\, dr} \qquad \ldots(4.1)$$

where

$$f_1 = \frac{\alpha E (3 - 2c)\, \bar{\theta}_0\, r^{n-1}}{y (n-1) D^n}$$

α is the coefficient of thermal expansion, E the Young's modulus and y the yield in tension.

It is found that the value $|\tau_{rr} - \tau_{\phi\phi}|$ is maximum at $r = a$, therefore yielding of the shell starts at the internal surface and eqn. (4.1) reduces to

$$|\tau_{rr} - \tau_{\phi\phi}| = \frac{p.\, a^{-3n+2c(n-1)} . \exp(f_1)}{2 \int_a^b r^{-3n+2c(n-1)-1} . \exp(f_1)\, dr} \equiv y_1 \qquad \ldots(4.2)$$

where y_1 is the yield stress and

$$f_1 = \frac{\alpha E (3 - 2c)\, \bar{\theta}_0\, a^{n-1}}{y (n-1) D^n}.$$

For incompressible material i. e. $(c \rightarrow 0)$ [see Seth equations (4.1) and (4.2) reduce to

$$\tau_{rr} = -p \frac{\int_r^b r^{-3n-1} . \exp(f_1)\, dr}{\int_a^b r^{-3n-1} . \exp(f_1)\, dr} \qquad \ldots(4.3)$$

$$\tau_{\phi\phi} = \tau_{zz} = \tau_{rr} + \frac{p r^{-3n} . \exp(f_1)}{2 \int_a^b r^{-3n-1} . \exp(f_1)\, dr}$$

and

$$y_1 = \frac{p\, a^{-3n}.\ \exp\,(f_1)}{2\int_a^b r^{-3n-1}.\ \exp\,(f_1)\, dr}$$

where

$$f_1 = \frac{3\alpha\ E\ \bar{\theta}_0\ a^{n-1}}{y\,(n-1)\ D^n}.$$

As a particular case, transitional creep stresses for a spherical shell under internal pressure only are obtained by putting $\theta_0 = 0$ in

equations (4.1) and (4.2) as

$$\tau_{rr} = -\,p\ \frac{[(b/r)^{3n-2c(n-1)} - 1]}{[(b/a)^{3n-2c\ (n-1)} - 1]}$$

$$\tau_{\phi\phi} = \tau_{zz} = p\ \frac{[\frac{1}{2}\{n\,(3-2c) - 2\,(1-c)\}\,(b/r)^{3n-2c\ (n-1)} + 1]}{[\ (b/a)^{3n-2c(n-1)} - 1]}\ . \quad ...(4.4)$$

These expressions are the same as obtained by Hulsukar[1]. For incompressible material, i. e. $c \rightarrow 0$, equations (4.4) become

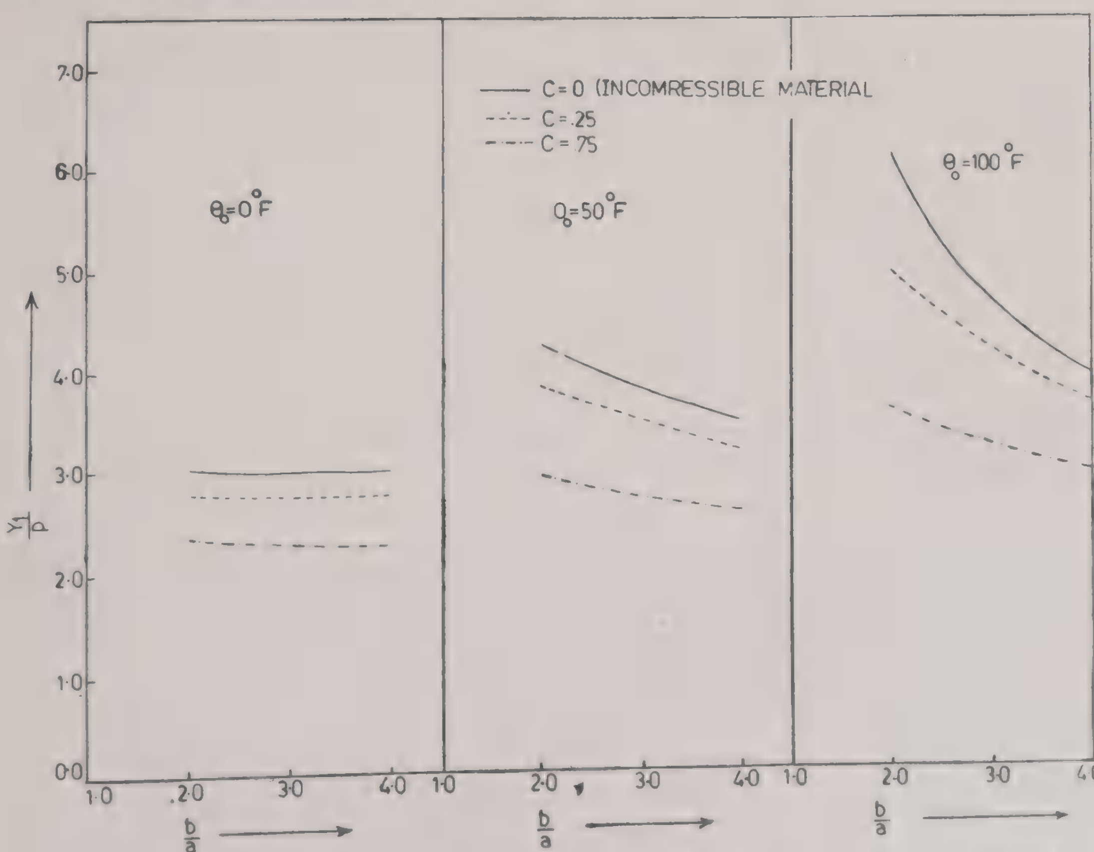

FIG. 1. Yielding ratio Y_1/p for various shell thickness ratios at different temperature for $n=2$.

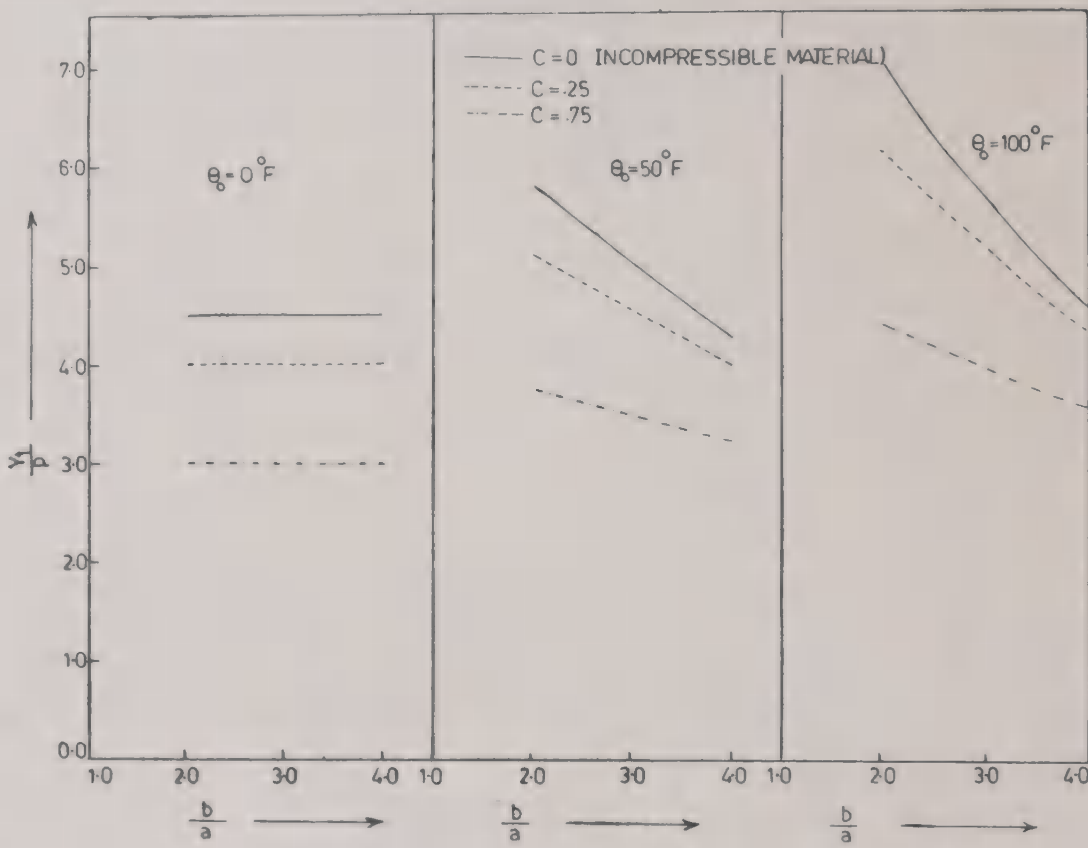

FIG. 2. Yielding ratio Y_1/p for various shell thickness ratios at different temperature for $n=3$.

$$\tau_{rr} = \frac{-p\,[(b/r)^{3n} - 1]}{[(b/a)^{3n} - 1]}$$

$$\tau_{\phi\phi} = \tau_{zz} = \frac{p\,[\frac{1}{2}(3n-2)(b/r)^{3n} + 1]}{[(b/a)^{3n} - 1]} \qquad \ldots(4.5)$$

These expressions are the same as given by Bailey[2] provided we put $n = 1/S$.

5. Numerical Illustration

To show the effect of combined pressure and temperature on a shell, this problem has been solved by using Simpson's rule for integration in eqns. (4.1), (4.2) and (4.3). For mild steel we take in various values as[15].

$$y = 3 \times 10^4 \text{ lb/in}^2,\; E = 3 \times 10^7 \text{ lb/in}^2 \text{ and } \alpha = 7.5 \times 10^{-6} \text{ per}^\circ \text{ F}.$$

In Figs. 1 2, curves have been drawn between yield y_1/p and different shell thickness ratios for $n = 2$ and 3 respectively. When heating effects are absent, it is seen that yielding of the thinner as well as thicker shells occurs generally at the same pressure, but with increasing temperature a thinner shell yields at higher pressure as

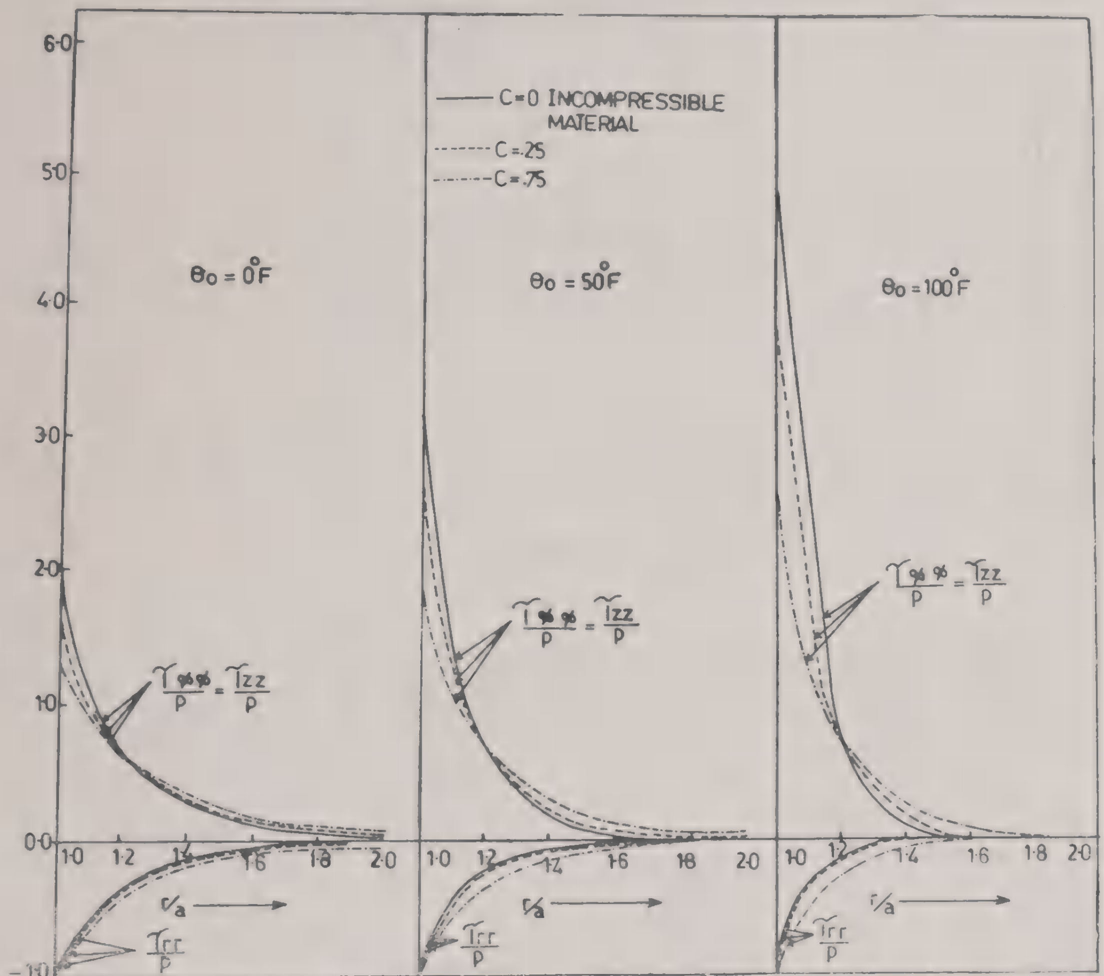

FIG. 3. Distribution of creep stresses due to temperature and pressure through wall of the shell for $n=2$.

compared to thicker shell. This yielding pressure goes on increasing with the increase in temperature and measure n. Shells made of incompressible material require higher pressure to yield as compared to shells made of compressible material. In Figs. 3 and 4 curves for radical and circumferential stresses have been drawn to show the combined effects of pressure and temperature for measure $n = 2$ and $n = 3$ respectively with respect to the ratio r/a. It has been found that the circumferential stress at the internal surface is higher for incompressible material than for compressible materials while at the outer surface the opposite situation occurs.

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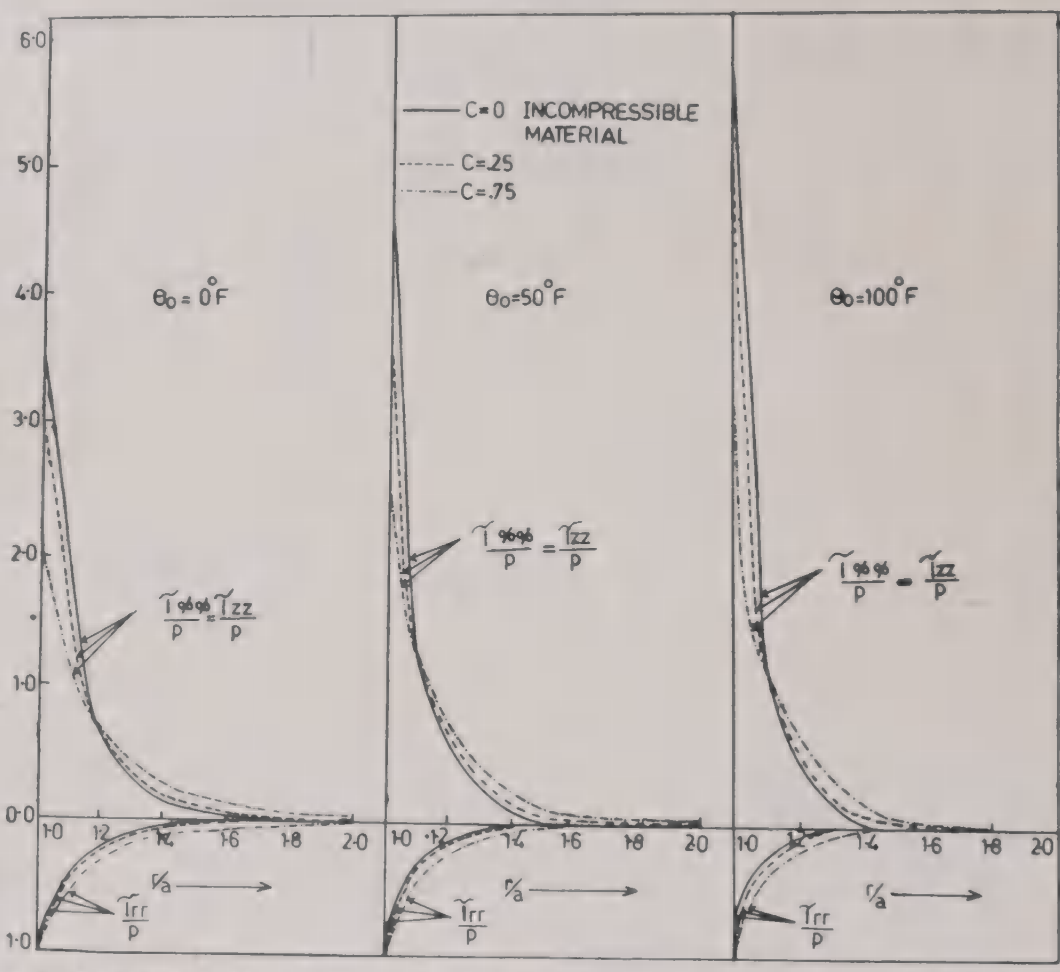

FIG. 4. Distribution of creep stresses due to temperature and pressure through wall of the shell for $n=3$.

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